Numerical relativity and the simulation of gravitational waves

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CEA, Bruyères-le-Chatel, 25 juin 2009
Gravitational waves: an introduction
**Gravitational waves**

**Definition**

Gravitational waves are predicted in Einstein’s relativistic theory of gravity: *general relativity*

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}
\]

Einstein’s equations

Linearizing around the flat (Minkowski) solution in vacuum \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \):

\[
\Box \left( h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \right) = -16\pi T_{\mu\nu}.
\]
Gravitational waves

Effects and amplitudes

The effect of a gravitational wave of (dimensionless) amplitude $h$ is a brief change of the relative distances

$$\frac{\Delta L}{L} \sim h.$$ 

Two polarization modes “+” and “×”: corresponding to the two dynamical degrees of freedom of the gravitational field.

Using the linearized Einstein equations:

$\Rightarrow$ at first order $h \sim \ddot{Q}$ (mass quadrupole momentum of the source), the total gravitational power of a source is

$$L \sim \frac{G}{c^5} s^2 \omega^6 M^2 R^4.$$
Gravitational waves: a laboratory experiment?

The proof of the existence of electromagnetic waves has been achieved by producing them in a laboratory and detecting them. Can we do this with gravitational waves?

- **Electromagnetic waves** are produced by accelerating **electric charges**;
- **Gravitational waves** are produced by accelerating **masses**.

Trying to accelerate a mass by rotating it:

Consider a cylinder made of steel:
- one meter in diameter and **twenty** meters long,
- weighting about **490 tons**,
- rotating at a maximal velocity of 260 rotations/minute (before breaking apart),

⇒ **Absolutely no hope** of detection, the emission is much too low.
Gravitational waves

Astrophysical sources

The problem stems from the constant factor in

$$L \sim \frac{G}{c^5} s^2 \omega^6 M^2 R^4$$

Introducing the **Schwarzschild radius** (radius of a black hole having the same mass) $$R_S = \frac{2GM}{c^2}$$, one gets

$$L \sim \frac{c^5}{G} s^2 \left( \frac{R_S}{R} \right)^2 \left( \frac{\nu}{c} \right)^6$$

⇒ accelerated masses:

- with strong gravitational field \(\leftrightarrow\) **compact**: neutron stars & black holes,
- at relativistic speeds,
- far from spherical symmetry \(s \lesssim 1\).

Binary systems of compact objects, neutrons stars & supernovae.
Gravitational waves

Ground detectors

LIGO: USA, Louisiana

LIGO: USA, Washington

VIRGO: France/Italy near Pisa

Michelson-type interferometers with 3 km (VIRGO) and 4 km (LIGO) long arms and almost perfect vacuum! Frequency range $10 \rightarrow 10000$ Hz.

⇒ Have been acquiring data together since a couple of years.
Gravitational waves

Space project LISA

On Earth, the vibrations propagating on the crust (seismic noise, human activities, ...) are limiting the detectors’ sensitivity.

⇒ LISA project (ESA / NASA) should be launched in 2019: 3 satellites at 5 millions kilometers one from another, in orbit around the Sun, 20 degrees behind the Earth. Frequency range $10^{-4} \rightarrow 1$ Hz.

Many more sources to be detected, with even a few certain ones.
The signal at the output of the detector
\[ \sigma(t) = h(t) + n(t), \]
with \( h(t) \leq n(t) \).

- The probability of detection is greatly enhanced in case of matched filtering: convolution with a priori known signal.

⇒ Need of full database of possible waveforms, to be computed by any means: analytic (post-Newtonian, ...) or numeric (our group).
Formulation of Einstein equations
**3+1 FORMALISM**

Decomposition of spacetime and of Einstein equations

**Evolution Equations:**

\[
\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_\beta K_{ij} =
- D_i D_j N + N R_{ij} - 2 N K_{ik} K^k_j + N \left[ K K_{ij} + 4 \pi ((S - E) \gamma_{ij} - 2 S_{ij}) \right]
\]

\[
K^{ij} = \frac{1}{2N} \left( \frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i \right).
\]

**Constraint Equations:**

\[
R + K^2 - K_{ij} K^{ij} = 16 \pi E,
\]

\[
D_j K^{ij} - D^i K = 8 \pi J^i.
\]

\[
g_{\mu\nu} \, dx^\mu \, dx^\nu = -N^2 \, dt^2 + \gamma_{ij} \left( dx^i + \beta^i \, dt \right) \left( dx^j + \beta^j \, dt \right)
\]
Constrained / free formulations

As in electromagnetism, if the constraints are satisfied initially, they remain so for a solution of the evolution equations.

**FREE EVOLUTION**

- start with initial data verifying the constraints,
- solve only the 6 evolution equations,
- recover a solution of all Einstein equations.

⇒ apparition of constraint violating modes from round-off errors. Considered cures:

- Using of constraint damping terms and adapted gauges.
- Solving the constraints at every time-step: e.g. fully-constrained formalism in Dirac gauge (2004).
**Conformal-Flatness Condition**

**Uniqueness issue**

4 constraints and the choice of time-slicing (gauge)

⇒ *elliptic system* of 5 non-linear equations can be formed

- Elliptic part of Einstein equations in the constrained scheme,
- Conformal-Flatness Condition (CFC): no evolution, no gravitational waves. used for computing initial data.

Because of non-linear terms, the elliptic system may not converge

⇒ the case appears for dynamical, very compact matter and GW configurations (before appearance of the black hole).
Summary of Einstein Equations

Constrained Scheme

**Evolution**

\[
\frac{\partial A^{ij}}{\partial t} = \nabla^k \nabla_k \tilde{\gamma}^{ij} + \ldots
\]
\[
\frac{\partial \tilde{\gamma}^{ij}}{\partial t} = 2N \Psi^{-6} A^{ij} + \ldots
\]

with

- \( \det \tilde{\gamma}^{ij} = 1 \),
- \( \nabla_j^{(f)} \tilde{\gamma}^{ij} = 0 \).

**Constraints**

\[
\nabla_j A^{ij} = 8\pi \Psi^{10} S^i,
\]
\[
\Delta \Psi = -2\pi \Psi^{-1} E - \Psi^{-7} \frac{A^{ij} A_{ij}}{8},
\]
\[
\Delta N \Psi = 2\pi N \Psi^{-1} + \ldots
\]
\[
A^{ij} = \Psi^{10} K^{ij}
\]

with

\[
\lim_{r \to \infty} \tilde{\gamma}^{ij} = f^{ij}, \quad \lim_{r \to \infty} \Psi = \lim_{r \to \infty} N = 1.
\]
Spectral methods for numerical relativity
How to deal with functions on a computer?

⇒ A computer can manage only integers

In order to represent a function $\phi(x)$ (e.g. interpolate), one can use:

- A finite set of its values $\{\phi_i\}_{i=0}^N$ on a grid $\{x_i\}_{i=0}^N$,
- A finite set of its coefficients in a functional basis

$$\phi(x) \simeq \sum_{i=0}^N c_i \Psi_i(x).$$

In order to manipulate a function (e.g. derive), each approach leads to:

- **Finite differences schemes**
  $$\phi'(x_i) \simeq \frac{\phi(x_{i+1}) - \phi(x_i)}{x_{i+1} - x_i}$$

- **Spectral methods**
  $$\phi'(x) \simeq \sum_{i=0}^N c_i \Psi'_i(x)$$
Convergence of Fourier Series

\[ \phi(x) = \sqrt{1.5 + \cos(x) + \sin^7 x} \]

\[ \phi(x) \simeq \sum_{i=0}^{N} a_i \Psi_i(x) \] with \( \Psi_{2k} = \cos(kx), \, \Psi_{2k+1} = \sin(kx) \)

\( N = 18 \)
Use of orthogonal polynomials

The solutions \((\lambda_i, u_i)_{i \in \mathbb{N}}\) of a singular Sturm-Liouville problem on the interval \(x \in [-1, 1]\):

\[-(pu')' + qu = \lambda w u,
\]

with \(p > 0, C^1, p(\pm 1) = 0\)

- are orthogonal with respect to the measure \(w\):

\[(u_i, u_j) = \int_{-1}^{1} u_i(x)u_j(x)w(x)dx = 0 \text{ for } m \neq n,\]

- form a spectral basis such that, if \(f(x)\) is smooth \((C^\infty)\)

\[f(x) \simeq \sum_{i=0}^{N} c_i u_i(x)\]

converges faster than any power of \(N\) (usually as \(e^{-N}\)).

Gauss quadrature to compute the integrals giving the \(c_i\)'s. Chebyshev, Legendre and, more generally any type of Jacobi polynomial enters this category.
**Method of weighted residuals**

General form of an ODE of unknown $u(x)$:

$$\forall x \in [a, b], \ Lu(x) = s(x), \text{ and } Bu(x)|_{x=a,b} = 0,$$

The approximate solution is sought in the form

$$\tilde{u}(x) = \sum_{i=0}^{N} c_i \Psi_i(x).$$

The $\{\Psi_i\}_{i=0...N}$ are called trial functions: they belong to a finite-dimension sub-space of some Hilbert space $\mathcal{H}_{[a,b]}$.

$\tilde{u}$ is said to be a numerical solution if:

- $B\tilde{u} = 0$ for $x = a, b$,
- $R\tilde{u} = L\tilde{u} - s$ is “small”.

Defining a set of test functions $\{\xi_i\}_{i=0...N}$ and a scalar product on $\mathcal{H}_{[a,b]}$, $R$ is small iff:

$$\forall i = 0 \ldots N, \quad (\xi_i, R) = 0.$$ 

It is expected that $\lim_{N \to \infty} \tilde{u} = u$, “true” solution of the ODE.
Inversion of linear ODEs

Thanks to the well-known recurrence relations of Legendre and Chebyshev polynomials, it is possible to express the coefficients \( \{b_i\}_{i=0}^{N} \) of

\[
Lu(x) = \sum_{i=0}^{N} b_i \begin{bmatrix} P_i(x) \\ T_i(x) \end{bmatrix}, \quad \text{with } u(x) = \sum_{i=0}^{N} a_i \begin{bmatrix} P_i(x) \\ T_i(x) \end{bmatrix}.
\]

If \( L = d/dx, x \times, \ldots, \) and \( u(x) \) is represented by the vector \( \{a_i\}_{i=0}^{N} \), \( L \) can be approximated by a matrix.

Resolution of a linear ODE

\[
\uparrow
\]

inversion of an \((N + 1) \times (N + 1)\) matrix

With non-trivial ODE kernels, one must add the boundary conditions to the matrix to make it invertible!
Some singular operators

\[ u(x) \mapsto \frac{u(x)}{x} \] is a linear operator, inverse of \( u(x) \mapsto xu(x) \).

Its action on the coefficients \( \{a_i\}_{i=0\ldots N} \) representing the \( N \)-order approximation to a function \( u(x) \) can be computed as the product by a regular matrix.⇒ The computation in the coefficient space of \( u(x)/x \), on the interval \([-1, 1]\) always gives a finite result (both with Chebyshev and Legendre polynomials).
⇒ The actual operator which is thus computed is

\[ u(x) \mapsto \frac{u(x) - u(0)}{x} . \]

⇒ Compute operators in spherical coordinates, with coordinate singularities

\[ \Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{\theta\phi} \]
**Explicit / implicit schemes**

Let us look for the numerical solution of \((L \text{ acts only on } x)\):

\[
\forall t \geq 0, \quad \forall x \in [-1, 1], \quad \frac{\partial u(x, t)}{\partial t} = Lu(x, t),
\]

with good boundary conditions. Then, with \(\delta t\) the time-step: \(\forall J \in \mathbb{N}, \quad u^J(x) = u(x, J \times \delta t)\), it is possible to discretize the PDE as

- \(u^{J+1}(x) = u^J(x) + \delta t \, Lu^J(x)\): **explicit time scheme** (forward Euler); easy to implement, fast but limited by the CFL condition.

- \(u^{J+1}(x) - \delta t \, Lu^{J+1}(x) = u^J(x)\): **implicit time scheme** (backward Euler); one must solve an equation (ODE) to get \(u^{J+1}\), the matrix approximating it here is \(I - \delta t \, L\). Allows longer time-steps but slower and limited to second-order schemes.
Multi-domain approach

Multi-domain technique: several touching, or overlapping, domains (intervals), each one mapped on $[-1, 1]$.

<table>
<thead>
<tr>
<th>Domain 1</th>
<th>Domain 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = -1$</td>
<td>$x_1 = 1$ $x_2 = -1$ $x_2 = 1$</td>
</tr>
<tr>
<td>$y = a$ $y = y_0$</td>
<td>$y = b$</td>
</tr>
</tbody>
</table>

- Boundary between two domains can be the place of a discontinuity $\implies$ recover spectral convergence,
- One can set a domain with more coefficients (collocation points) in a region where much resolution is needed $\implies$ fixed mesh refinement,
- 2D or 3D, allows to build a complex domain from several simpler ones,

Depending on the PDE, matching conditions are imposed at $y = y_0 \iff$ boundary conditions in each domain.
MAPPINGS AND MULTI-D

In two spatial dimensions, the usual technique is to write a function as:

\[
f : \hat{\Omega} = [-1, 1] \times [-1, 1] \to \mathbb{R}
\]

\[
f(x, y) = \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} c_{ij} P_i(x) P_j(y)
\]

The domain \( \hat{\Omega} \) is then mapped to the real physical domain, through some mapping \( \Pi : (x, y) \mapsto (X, Y) \in \Omega \).

\( \implies \) When computing derivatives, the Jacobian of \( \Pi \) is used.

COMPACTIFICATION

A very convenient mapping in spherical coordinates is

\[
x \in [-1, 1] \mapsto r = \frac{1}{\alpha(x - 1)},
\]

to impose boundary condition for \( r \to \infty \) at \( x = 1 \).
**Example:**

**3D Poisson equation, with non-compact support**

To solve \( \Delta \phi(r, \theta, \varphi) = s(r, \theta, \varphi) \), with \( s \) extending to infinity.

- setup two domains in the radial direction: one to deal with the singularity at \( r = 0 \), the other with a compactified mapping.

- In each domain decompose the angular part of both fields onto spherical harmonics:

\[
\phi(\xi, \theta, \varphi) \simeq \sum_{\ell=0}^{\ell_{\text{max}}} \sum_{m=-\ell}^{\ell} \phi_{\ell m}(\xi) Y_{\ell m}(\theta, \varphi),
\]

\( \forall (\ell, m) \) solve the ODE:

\[
\frac{d^2 \phi_{\ell m}}{d\xi^2} + 2 \frac{d \phi_{\ell m}}{\xi} - \frac{\ell (\ell + 1) \phi_{\ell m}}{\xi^2} = s_{\ell m}(\xi),
\]

match between domains, with regularity conditions at \( r = 0 \), and boundary conditions at \( r \to \infty \).
Application to binary compact stars
Inspiralling binaries

Astrophysical scenario: binary systems of compact objects evolve toward the final coalescence by emission of gravitational waves and angular momentum loss.

Stiff problem: the orbital and coalescence timescales are very different.

Post-Newtonian (perturbative) computations assume point-mass particles ⇒ valid until separation is comparable to size.

⇒ numerical simulation of initial data and evolution.
Binary neutron stars

Initial data:
irrotational flow
and
conformal-flatness approximation,
two adapted-grid system, to take into account tidal effects,

- use of realistic equations of state for cold nuclear matter,
- exploration of strange-quark equations of state.

Binary quark stars
**Binary black hole**

Stellar masses (for VIRGO) or galactic masses (for LISA).

- First realistic initial data (2002), with excision techniques,
- Good agreement with post-Newtonian computations,
- Determination of the **last stable orbit**, important for gravitational wave data analysis.
Stellar core-collapse simulations
Simplified physical model of core-collapse

The phenomenon of *supernova* is too rich to be fully-modeled on a computer

- relativistic hydrodynamics \((v/c \sim 0.3)\), including shocks, turbulence and rotation,
- strong gravitational field \(\Rightarrow\) General Relativity?
- neutrino transport (matter deleptonization)
- nuclear equation of state (EOS)
- radiative transfer and ionization of higher layers
- magnetic field?

\(\Rightarrow\) to track gravitational waves, some features must be neglected...and we use an effective model (not trying to make them explode).
**Simplified physical model of core-collapse**

- General-relativistic hydrodynamics: 5 hyperbolic PDEs in conservation form,
- Conformal-flatness condition for the relativistic gravity: 5 elliptic PDEs to be solved at each time-step,
- Initial model is a rotating polytrope with an effective adiabatic index $\gamma \lesssim 4/3$. During the collapse, when the density reaches the nuclear level, $\gamma \rightarrow \gamma_2 \gtrsim 2$,
- Passive magnetic field,
- Lepton fraction deduced from density, following spherically-symmetric simulations with more detailed neutrino transport.
Combination of two numerical techniques

- hydrodynamics ⇒ High-Resolution Shock-Capturing schemes (HRSC), also known as Godunov methods, here implemented in General Relativity;

- gravity ⇒ multi-domain spectral solver using spherical harmonics and Chebyshev polynomials, with a compactification of type $u = 1/r$.

Use of two numerical grids with interpolation:

- matter sources: Godunov (HRSC) grid → spectral grid;

- gravitational fields: spectral grid → Godunov grid.

First achieved in the case of spherical symmetry, in tensor-scalar theory of gravity (JN & Ibáñez 2000). Spares a lot of CPU time in the gravitational sector, that can be used for other physical ingredients.
Toward a realistic relativistic collapse

Together with the use of a purely finite-differences code in full GR, first results of realistic collapse of rotating stellar iron cores in GR

- with finite temperature EOS;
- (approximate) treatment of deleptonization.

⇒ complete check that CFC is a good approximation in the case of core-collapse.
**Neutron star oscillations**

Study of non-linear axisymmetric pulsations of rotating relativistic stars

- uniformly and differentially rotating relativistic polytropes $\Rightarrow$ differential rotation significantly shifts frequencies to smaller values;
- mass-shedding-induced damping of pulsations, close to maximal rotation frequency.

- most powerful modes could be seen by current detectors if the source is about $\sim 10$ kpc;
- if 4 modes are detected, information about cold nuclear matter equation of state could be extracted $\Rightarrow$ gravitational asteroseismology.
**SUMMARY – PERSPECTIVES**

- Numerical simulations of sources of gravitational waves are of highest importance for the detection.
- Use of spectral methods can bring high accuracy with moderate computational means (exploration of parameter space).
- Spectral methods can be associated with other types, as in the core-collapse code presented here.
- Core-collapse code: going beyond conformal-flatness approximation ⇒ better extraction of waves.
- Improvement of this code: realistic EOS, temperature effects for very massive star collapses (hypernovae).
- Neutrinos? Ongoing work with M. Oertel.
- Study of the electro-weak processes: electron capture rate, nucleon effective masses and EOS. Work by A. Fantina, P. Blottiau, J. Margueron, P. Pizzochero, …