Fractals and nonstandard analysis

L. Nottale and J. Schneider
Observatoire de Meudon, 92195 Meudon Principal Cédex, France

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We describe and analyze a parametrization of fractal "curves" (i.e., fractal of topological dimension 1). The nondifferentiability of fractals and their infinite length forbid a complete description based on usual real numbers. We show that using nonstandard analysis it is possible to solve this problem: A class of nonstandard curves (whose standard part is the usual fractal) is defined so that a curvilinear coordinate along the fractal can be built, this being the first step towards the possible definition and study of a fractal space. We mention fields of physics to which such a formalism could be applied in the future.

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I. INTRODUCTION

The concept of fractals, introduced by B. Mandelbrot,\textsuperscript{1-3} applies to any curve, object, or set "whose form is extremely irregular and/or fragmented at all scales." More precisely, let \( D \) be the fractal dimension (i.e., the Hausdorff-Besicovitch dimension); a fractal is defined\textsuperscript{1,2} as a set for which \( D \) is greater than the topological dimension \( D_T \).

Nowadays this concept is increasingly considered in physics for several reasons:

(i) Fractal curves are functions which are continuous but nowhere differentiable; this property has already been observed for some natural phenomena, such as particle trajectories in quantum mechanics.\textsuperscript{4,5}

(ii) The length of a fractal curve is dependent on the resolution with which it is measured and diverges when the resolution tends to be infinite.

(iii) A fractal dimension can be any real number, so this concept may apply to fields of physics such as critical behavior phenomena, where noninteger dimension has become a necessity.

Mandelbrot\textsuperscript{1-3} pointed out many examples of the contribution fractals can bring to the description of natural phenomena such as the length of a coastline, the distribution of matter in the universe, turbulence, moon craters... Furthermore, the concept of Hausdorff dimension has been applied to QCD jets,\textsuperscript{6} gauge theories,\textsuperscript{7} critical behavior,\textsuperscript{8} fluctuations of the early universe,\textsuperscript{9} or quantum-mechanical pathways.\textsuperscript{10}

However, in most cases, the authors limit themselves to the calculation of a fractal dimension or use fractals in a purely descriptive way (but see Le Mhauté \textit{et al.}\textsuperscript{10,11}). Because of the wide domain where phenomena seem to exhibit a fractal behavior, one is entitled to wish that a more thorough use of this concept would be possible, e.g., by building a formalism based on fractals and suitable for physics. In fact, no explicit calculation is presently possible on the fractal \textit{itself} (i.e., the limit object instead of one of its approximations). It is the aim of this paper to show that nonstandard analysis, as built up by Robinson\textsuperscript{12} is well adapted to such calculations.

In this paper, we first parametrize fractal "curves" (i.e., fractals of topological dimension 1) in the Cesaro\textsuperscript{13} way (Sec. II). Then some paradoxical properties of fractals are evidenced and are clarified by the use of nonstandard analysis (Sec. III) as a way to build intrinsic curvilinear coordinates along a fractal curve (Sec. IV). This is hopefully a first step towards the definition of a fractal space by its own, while so far fractals have been considered as subsets of an integer-dimensional space.

II. PARAMETRIC EQUATION OF A FRACTAL CURVE

Consider a generalized von Koch curve in the \( \mathbb{R}^2 \) (or \( \mathbb{C} \)) plane. It can be built from an initial curve \( F_1 \) made up of \( p \) segments of equal length \( 1/q \) which connect the origin to the point \([0,1]\) [see Fig. 1]. Let \( \phi_j \), be the polar angle of the \( j \)th segment and \( Z_j = X_j + iY_j = (1/q)^j \sum_{k=0}^{j-1} e^{i\phi_k} \) the complex coordinate of a breaking point \( P_j \). Two conditions hold between these data:

\[
\begin{align*}
\sum_{k=0}^{j-1} e^{i\phi_k} = q, & \quad Z_{j+1} - Z_j = \frac{1}{q} e^{i\phi_j}.
\end{align*}
\]

A curve \( F_2 \) is obtained by substituting each segment of \( F_1 \) by \( F_2 \) itself, scaled at its length \( 1/q \), as illustrated in Fig. 2. The resulting curve of an infinite sequence of these steps (substitution of each segment of \( F_n \) by \( q^{-n} F_1 \), giving \( F_{n+1} \)) is the fractal \( F \).

As indicated by Mandelbrot\textsuperscript{1} in the case of the Peano curve, \( F \) can be parametrized by a real number \( x \in [0,1] \) developed in the counting base \( p \) in the form (see Fig. 2):

\[
x = 0.x_1x_2..., \quad x_n = \sum_{k=1}^{\infty} x_k p^{-k}.
\]

The fractal will apparently be completely defined when the complex coordinate \( Z(x) \) of the point on \( F \) parametrized by \( x \) is known. \( Z(x) \) can be easily obtained, thanks to the above building process of \( F \), under the form

![FIG. 1. Building of the basic structure of a fractal curve.](image-url)
We first recall that, with its fractal dimension $D = \log p/\log q$, the length of the fractal is infinite and its surface vanishes since the $F_i$ length and surface are

$$\mathcal{L}_n / \mathcal{L}_0 = (p/q)^n = q^{n-\beta} - 1,$$

$$\mathcal{S}_n / \mathcal{S}_0 = (p/q)^n = q^{-2n - D}.$$  

(2.9)

Moreover, however small the difference of parameters $x(2) - x(1)$ for two points $M_2$ and $M_1$ on the fractal, though the distance in the plane $|Z(x_2) - Z(x_1)|$ vanishes, the distance along the fractal remains infinite.

However, let us build the following sequence:

$$a_n = a^{(n-1)} = (aq^n)^{p-q} = 0.a_{n1}a_{n2} \ldots a_{np},$$  

(2.10)

where $a$ is a nonzero number. Assume now that the $a_n$'s are the parameters of a given sequence of points $M_n$ on the curves $F_n$. The curvilinear coordinates of these points along $F_n$ are equal to

$$l_n = \mathcal{L}_n a_n = q = \text{const.}$$  

(2.11)

For example, in Fig. 4 the sequence of parameters $0.1, 0.03, 0.021, 0.0123, \ldots$ defines points at a constant distance 0.1 from the origin on the respective curves $F_1, F_2, F_3, F_4, \ldots$. Let now $n \to \infty$ and consider the limit point $M$ of the sequence of points $M_n$ on $F$. From Eq. (2.10) its parameter is $\lim (a_n) = 0$, i.e., $M$ coincides with the origin $O$; however, from Eq. (2.11), the curvilinear distance between $O$ and $M$ on the fractal is $a \neq 0$.

From this “paradox” we conclude that the real coordinate $x$ is insufficient to describe thoroughly the fractal curve $F$; the distance along $F$ between two points parametrized by two different $x$'s is infinite, while points separated by a finite distance along $F$ correspond to the same values of $x$. Thus another formalism is needed, and more precisely a set “larger” than $\mathbb{R}$ Nonstandard analysis 12 which allows dealing both with infinitesimals and infinite numbers, provides such a frame, well adapted to the study of fractals, as we show in the following sections.

III. NONSTANDARD ANALYSIS: A REMINDER

Nonstandard analysis (NSA) may be considered as the solution A. Robinson worked out for the old problem of infinitesimals. Leibnitz, founder of differential calculus, thought of “infinitely small” and “infinitely large” numbers as ideal numbers to which operations on usual numbers

\begin{align*}
\alpha^+ + \beta^- &< \pi - \delta \omega_j, \quad \alpha^- + \beta^+ < \pi + \delta \omega_j, \\
\gamma^- &< \delta \omega_j < \gamma^+.
\end{align*}

Let us now illustrate the nonstandard character of fractals, answering (by the negative) the question: Do Eqs. (2.2) and (2.6) characterize all properties of $F$ in the $\mathbb{R}^2$ plane?

\begin{itemize}
  \item $F_1$
  \item $F_2$
  \item $F_3$
  \item $F_4$
\end{itemize}

\begin{itemize}
  \item $a_1 = 0.1$
  \item $a_2 = 0.03$
  \item $a_3 = 0.021$
  \item $a_4 = 0.0123$
\end{itemize}

FIG. 4. A sequence of points on $F_i$ with constant curvilinear distance to the origin.


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would apply, though he was unable to build a coherent system: In fact the up to now accepted significant of these expressions involves the theory of limits and the so-called epsilon-delta method according to the works of Cauchy and Weierstrass. However, Robinson\(^{12,14}\) demonstrated that real numbers \(\mathbb{R}\) can be extended to \(*\mathbb{R}\) which contains infinitely small and infinitely large numbers.

We will not try here to present a detailed description of NSA, but only recall some basic results useful in what follows independent of the precise way the theory is evolved, using, e.g., free ultrafilters and equivalence classes of sequences; see Robinson,\(^{14}\) Stroyan and Luxemburg,\(^{15}\) or using an axiomatic extension of the Zermelo set theory, see Nelson.\(^{16}\)

The set \(\mathbb{R}\) of hyper-real numbers is a totally ordered and non-Archimedean field. The set \(\mathbb{R}\) of standard numbers is a subset of \(*\mathbb{R}\). \(*\mathbb{R}\) contains infinite elements, i.e., elements \(A\) such that \(\forall n \in \mathbb{N}, |A| > n\). It also contains infinitesimal elements, i.e., B such that \(\forall n \in \mathbb{N} (n \neq 0),|B| < 1/n\). A finite element C is defined: \(\exists m \in \mathbb{N}, |C| < m\). The set of infinitesimals is denoted by \(\alpha\), the set of finite numbers by \(\mathbb{D}\) and the set of infinite numbers by \(*\mathbb{D}\). Any finite number \(a \in \mathbb{D}\) can be split up in a single way as \(a = r + e\), where \(r \in \mathbb{R}\) and \(e \in \alpha\).

In other words, the hyper-realss contain the ordinary reals with new numbers \(a\) clustered infinitesimally closely around each ordinary real \(r\). Their set \(|\alpha|\) is called the monad of \(r\). The real \(r\) is said to be the "standard" part of the hyper-real \(a\), a function denoted by \(\text{st}(a) = r\). The "st" function is very useful for nonstandard demonstrations of standard theorems. For instance a sum \(\sum_{\alpha} f_{\alpha}\) is said to converge if for different \(\alpha\)'s belonging to the set of infinite hypernatural numbers \(*\mathbb{N}_\omega\), \(\text{st}(\sum_{\alpha} f_{\alpha})\) are all equal to the same finite number. Apart from the strict equality \(\sim\), one introduces the equivalence relations \(\sim^+\) meaning "infinitely close to," i.e., \(a \sim b (\sim^+) \iff \text{st}(a - b) = 0\), and the relation \(\sim^\sim\) meaning "of the order of," i.e., \(a \sim b (\sim^\sim) \exists k \in \mathbb{R}, k \neq 0\) such that \(a = kb\).

Formal descriptions of NSA may be found in Refs. 14-17. There have also been some attempts of applications of NSA to physics, e.g., Kelemen and Robinson,\(^{18,19}\) Moore,\(^{20}\) Anderson.\(^{21}\)

IV. NONSTANDARD COORDINATES ALONG FRACTALS

In Sec. II, a fractal in \(\mathbb{R}^2\) was parametrized by a real number belonging to the interval \([0,1]\):

\[
x = \frac{x_1}{p^1} + \frac{x_2}{p^2} + \ldots + \frac{x_n}{p^n} + \ldots
\]

(4.1)

Let us generalize the usual fractal by introducing a curve \(F_{\omega}\) in \(*\mathbb{R}^2\), parametrized by an hyper-real number \(x^*\epsilon[0,1]\), as defined by the *finite power series expansion

\[
x^* = \frac{x_1}{p} + \ldots + \frac{x_n}{p^n} + \ldots + \frac{x_{\omega}}{p^{\omega}} + \ldots = \sum_{n=1}^{\omega} \frac{x_n}{p^{n}}
\]

(4.2)

where \(\omega \in *\mathbb{N}_\omega\). In other terms, \(F_{\omega}\) may be obtained by applying the building process in Sec. II (i.e., build \(F_{n+1}\) by substituting to each segment of \(F_n\), \(F_1\) scaled by \(q^{-n}\) \(\omega\) times. It should be noticed that \(F_{\omega}\) is not a fractal in the nonstandard sense (since the fragmentation is * limited up to \(\omega\)) but its standard part is identical to the usual fractal, i.e.,

\[
F = \text{st}(F_{\omega})
\]

Then the study of \(F_{\omega}\) allows us to study the properties of \(F\), thanks to the standardization axiom.

A first advantage is that, while the length of \(F\) was undefined, the length of \(F_{\omega}\) is defined:

\[
L_{\omega} = \left(\frac{p}{q}\right)^{\omega} = q^{-\omega(p-1)}
\]

(4.3)

While the surface of \(F\) was zero, the surface of \(F_{\omega}\) is an infinitesimal:

\[
S_{\omega} = \left(\frac{p}{q}\right)^{2\omega} = q^{-2(1-\omega)}
\]

(4.4)

The curvilinear coordinate \(x^*\) of the point parametrized by \(x^*\) is now also defined on \(F_{\omega}\):

\[
x^* L_{\omega} = q^{-\omega}|x_\omega + x_{\omega-1} p + \ldots + x_1 p^{\omega-1}|
\]

(4.5)

This verifies that \(F_{\omega}\) is built up by elementary segments of length \(q^{-\omega}\). By using an infinitely great magnifying power, \(F_{\omega}\) can be drawn exactly (while this was not the case for \(F\)) as in Fig. 5. The fractal is no more a limit concept.

Let us now utilize the new concept of \(F_{\omega}\) to study or clarify some problems specific of fractals, which may be relevant for physical applications.

A. Finite distance along the fractal

In Sec. II we obtained a point separated from the origin by a finite nonzero distance along the fractal, its parameter was \(x = 0\). This situation may be clarified by defining \(M\) such that

\[
\hat{x}_M = L_{\omega} \omega^{-M} = 1
\]

(4.6)

Then \(M\) is a solution of the equation

\[
p^M - M = q^n
\]

(4.7)

so that

\[
M = a(1 - \log p / \log q) = a(1 - 1/D)
\]

(4.8)

Generally \(M\) is not an integer; therefore we define \(\Lambda \in *\mathbb{N}_\omega\) as

\[
\Lambda = \text{Int}(M) = \text{Int}(a(1 - 1/D))
\]

(4.9)

where \(\text{Int}(X)\) is the integer part of \(X\) (it is straightforward to

FIG. 5. Infinite magnification of curve \(F_{\omega}\), the standard part of which is the fractal \(F\).
verify that this function is still defined in \( *R \). For any \( \lambda \), such that \( \lambda \neq \lambda \) (the relation \( \sim \) of the order of) has been defined in Sec. III. The curvilinear distance along the fractal \( \xi_{i,\lambda} \), \( = \xi_{i,\lambda} \) \( \neq \lambda \) belongs to \( R \). However, this allows one to understand why \( x = 0 \) while \( \xi \neq 0 \), the corresponding distance in the \( R^3 \) plane is an infinitesimal, \( q^{-1} \), the standard part of which is thus zero.

Consider two points on \( F_{\omega} \) separated by a curvilinear distance \( \xi \). Depending on the power \( n \) of \( p \) in the expression \( 4.6 \) of \( \xi \), three levels may be distinguished on \( F_{\omega} \):

- \( \omega \in N \): Finite distance in the \( R^3 \) plane, infinite along the fractal.
- \( \omega \sim \lambda \): Infinitesimal distance in the \( R^3 \) plane, finite distance along the fractal.
- \( \omega \sim \omega \): Infinitesimal distance in \( R^2 \) and on \( F_{\omega} \). At this level the two distances are of the same order.

B. Intrinsic building of \( F \)

Parametric equations for \( F \) have been given in Sec. II (Eq. 2.6): It relates the parameter \( x \) to the coordinates \( X, Y \) in the \( R^3 \) plane. An intrinsic building of \( F_{\omega} \) independent of the plane in which it is embedded, is possible: We only need to know the change in direction from one elementary segment of length \( q^{-1} \) to the following one. In Sec. II, we had set \( \omega_{\theta} = \omega_{\theta - 1} \), the angle between two segments in \( F_1 \) (see Fig. 1). Consider the infinitesimal segment of curvilinear coordinate \( \xi \) on \( F_{\omega} \), and define \( h \) such that \( x_{\omega_{\theta - 1}} \) is the first nonzero figure of the hypernatural number \( q^{-1} \) in the base \( p \):

\[
\xi = q^{-1} \times [0 + 0 + p + \ldots + 0 + p^{h-1} + x_{\omega_{\theta - 1}} p^{h} + \ldots + x_{\theta} p^{\theta - 1}].
\]

The relation for the angle we were looking for and which allows an intrinsic building of \( F_{\omega} \) is simply given by

\[
\delta \phi_{\omega} = \delta \phi_{\omega_{\theta - 1}}.
\]

(4.12)

A somewhat paradoxical property of fractals is that this relative angle exists and may be computed for any value of \( \xi \), while on the contrary the absolute angle \( \omega_{\theta} (\xi) \) does not exist as soon as \( \xi q^{\theta} \) becomes infinite, since

\[
\omega(\xi) = \sum_{\theta=1}^{\xi} \omega_{\theta} (\xi)
\]

(4.13)

so that the value of \( \delta \phi_{\omega} (\xi) \) depends on that of \( \omega (\xi) \) (this is the nonstandard transfer of the nondifferentiability of \( F \)).

C. Family of curves \( F_{\omega} \) and \( \epsilon \) differentiability

The standard part of any curve differing from \( F_{\omega} \) only by infinitesimals will also be the fractal \( F \). This is true for any \( F_{\omega} \) with \( \omega \in *N \), \( \omega \neq \omega \), but might be generalized. In particular the "broken" aspect of \( F_{\omega} \) can be given up and \( F_{\omega} \) replaced by a smoothed nonstandard curve \( F_{\omega} \), with \( F = st(F_{\omega}) \). A kind of differentiability can be defined for \( F_{\omega} \).

Indeed, nonstandard analysis may be used to define differentiability of standard functions.
sion $D$ have been discussed. However the fractal dimension can clearly be a local property, varying with the curvilinear coordinate $D = D(\xi)$; this is more easily understandable by remembering the underlying infinitesimal structure.

In the future, we will try to extend the parametrization presented here to fractals of topological dimensions greater than one. Instead of studying fractal objects embedded in an Euclidean space, the aim of such a work would be to define a fractal space intrinsically. Its dimension could then be a generic parameter and function of the coordinates in the same way as curvature occurs for curved spaces.

In a forthcoming paper, one of the authors will consider physical applications of this formalism to quantum mechanics (uncertainty relations and theory of measure). We think the notions developed here could help in the field of quantization of gravity: the studies in this field usually assume without question an underlying, eventually foamy, $4$-manifold. Below some characteristic length, space itself could become a fractal. To be more specific one could for instance perform latticelike gauge field calculations on a fractal. Let us notice that the path-integral ingredient of such calculations has also recently been formulated in terms of nonstandard analysis. When applied to cosmology these ideas lead to the natural speculation that the very early universe experienced in its whole a noninteger-dimensional phase. To conclude we hope that these new trends will help to answer the question: Why does the macroscopic space appear to be three dimensional?

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