The pressure tensor in tangential equilibria

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Abstract. The tangential equilibria are characterized by a bulk plasma velocity and a magnetic field that are perpendicular to the gradient direction. Such equilibria can be spatially periodic (like waves), or they can separate two regions with asymptotic uniform conditions (like MHD discontinuities). It is possible to compute the velocity moments of the particle distribution function. Even in very simple cases, the pressure tensor is not isotropic and not gyrotropic. The differences between a scalar pressure and the pressure tensor are significant when the gradient scales are of the order of the Larmor radius; they concern mainly the ion pressure tensor.

Key words. Magnetospheric physics (magnetopause, cusp and boundary layers) – Space plasma physics (discontinuities; kinetic and MHD theory)

1 Introduction

We define tangential equilibria as monodimensional equilibrium structures, where the magnetic field is perpendicular to the gradient direction and the plasma velocity along the gradient direction is null. When the magnetic field does not keep a constant direction, these solutions are sometimes referred to as sheared equilibria. These equilibria can be described by MHD or multifluid theories (for example, as tangential discontinuities), but if the gradients are sharp in comparison to one ion Larmor gyroradius, the fluid models cannot look inside these structures.

Sharp tangential equilibria are often met in space collisionless plasmas. A solar wind tangential discontinuity was crossed by the Cluster spacecraft and its thickness was estimated to 600–1000 km (Dunlop et al., 2002). The magnetic field amplitude was 30 nT; estimating an ion temperature of 100 eV, the thickness of the tangential discontinuity was less than three ion Larmor radii. The magnetopause is sometimes found to be a tangential discontinuity, it generally has a similar thickness. Tangential discontinuities exist also inside the Earth’s magnetosphere: tangential current sheet crossings by Cluster allowed an estimated thickness of 1126 km, and even 400 km (Petrukovich et al., 2003), that is, of the order of the ion Larmor radius. Plasma cavities in the auroral zone and in the solar wind have a tangential equilibria geometry, and the density gradient sometimes do not exceed 1.4 km, that is, two or three ion Larmor radii (Hilgers et al., 1992).

Theoretical works on tangential equilibria have shown the existence of analytical isothermal (Harris, 1962; Channell, 1976), or non isothermal solutions (Attico and Pegoraro, 1999). Other works, mainly focused on the study of the Earth’s magnetopause and reviewed by Roth et al. (1996), have shown solutions that satisfy a larger class of constraints, but where the differential equations are solved numerically. Another class of tangential equilibria (Mottez, 2003) can explain the non isothermal equilibrium of deep plasma cavities in the Earth’s auroral zone. All these equilibria have been developed in the frame of the collisionless plasma kinetic theory; they are solutions of the Maxwell-Vlasov equations. They all suppose particle distribution functions given, for each species s by

\[ f_s \left( \alpha_s \right) = \int_{a_1}^{a_2} da \left( \frac{\alpha_s}{\pi} \right)^{3/2} e^{-\frac{r^2}{\alpha_s}} G_{\alpha_s}(p_y, p_z), \]  

(1)

where \( a \) is a scalar referring to an isothermal (trapped or passing) particle population, \( a_1 \) and \( a_2 \) are arbitrary (sometimes infinite), and \( a_{\alpha_s}=(m_s/2T_{\alpha_s})^{1/2} \). The choice \( G_{\alpha_s} \) is specific to each solution (Mottez, 2003). The variables

\[ p_y = v_y + \frac{q}{m} A_y(x) = v_y + \frac{q}{m} A(x) \cos \theta(x) \]

\[ p_z = v_z + \frac{q}{m} A_z(x) = v_z + \frac{q}{m} A(x) \sin \theta(x) \]

\[ E = v_x^2 + v_y^2 + v_z^2 - \frac{2q}{m} \Phi(x) \]  \( \Phi \)

are the invariants of the particle motion in the monodimensional equilibrium (\( \partial_y=\partial_z=0 \), \( \partial_t=0 \)).

Many studies have been devoted to the stability of these equilibria. They are generally based on MHD, Hall-MHD, or multifluid equations. What do we lose when we jump from the kinetic Vlasov theory to the fluid approach? We shall investigate this question through an evaluation of the fluid moments of the tangential equilibria; we will concentrate especially on the pressure tensor.
2 The basic equations of tangential equilibria

We consider monodimensional equilibria, and they are the directions of invariance. In tangential equilibria $B_y(x) = 0$ but $B_z(x)$ and $B_v(x)$ are functions of $x$. As $\partial_y = 0$, the electric field derives from a scalar potential $\Phi(x)$, and only $E_z(x)$ can be different from zero. The vector potential has two components $A_y(x), A_z(x)$, and

$$B_z(x) = d_x A_y,$$
$$B_v(x) = -d_x A_z. \tag{3}$$

Since the particle distribution functions given in Eq. (1) depend only on the invariants of the motion of individual particles, they are solutions of the Vlasov equation. If $T_{as}$ is independent of $a$, the equilibrium is isothermal and $T_{as}$ is the temperature of the species $s$. We consider plasmas formed of an electron population ($s=e$) and one ion species ($s=i$). In order not to overload the equations, we express the dependence of the parameters on $s$ only when two different species are treated in the same equation.

The equilibria must also verify the Maxwell equations. For tangential equilibria, they are particularly simple. The Ampere condition is

$$J_y = -\frac{1}{\mu_0} d_x B_z = -\frac{1}{\mu_0} d_x^2 A_y \tag{4}$$
$$J_z = \frac{1}{\mu_0} d_x B_v = -\frac{1}{\mu_0} d_x^2 A_z.$$

The contribution of each species to the current density $J_y$ is

$$J_y(x) = q \int dv v_y f,$$ \tag{5}

with a similar equation for $J_z$. The dependence of $J_y$ in $x$ comes from the dependence of $f$ on $E_y, p_y$, and $p_z$, which themselves depend on the scalar and vector potentials $\Phi(x), A_y(x)$ and $A_z(x)$. The Poisson equation writes

$$\frac{d^2 \Phi(x)}{dx^2} = -\frac{e}{\epsilon_0} (n_i(x) - n_e(x)), \tag{6}$$

where

$$n_i = \int dv f_i \quad \text{and} \quad n_e = \int dv f_e. \tag{7}$$

3 The first velocity moments of the distribution function

We can express the total energy $E$ in Eq. (1) as the sum of the electric and kinetic energy:

$$f = \int_{a_1}^{a_2} da \left( \frac{a_d}{\pi} \right)^{3/2} e^{-\frac{2aq\Phi}{m}a^2} e^{-a_e x^2} G_a(p_y, p_z). \tag{8}$$

Eliminating $v_y$ and $v_z$:

$$f = \int_{a_1}^{a_2} da \left( \frac{a_d}{\pi} \right)^{3/2} e^{-\alpha_a x^2} \left\{ e^{-\frac{2aq\Phi}{m}a^2} + (p_z - \frac{q}{m} A_z)^2 \right\} \times$$
$$G_a(p_y, p_z) e^{-\frac{2aq\Phi}{m}a^2}. \tag{9}$$

For a given particle species, the particle density is

$$n(x) = \int_{a_1}^{a_2} da \left( \frac{a_d}{\pi} \right) \int e^{-\alpha_a [(p_y - \frac{q}{m} A_y)^2 + (p_z - \frac{q}{m} A_z)^2]} \times$$
$$G_a(p_y, p_z) e^{-\frac{2aq\Phi}{m}a^2} dp_y dp_z, \tag{10}$$

where $p_y$ and $p_z$ vary from $-\infty$ to $+\infty$. Let us define $n_a$ that depends on $x$ through the potentials $A_y(x), A_z(x)$, and $\Phi(x)$,

$$n_a(x) = \left( \frac{a_d}{\pi} \right) \int e^{-\alpha_a [(p_y - \frac{q}{m} A_y)^2 + (p_z - \frac{q}{m} A_z)^2]} \times$$
$$G_a(p_y, p_z) e^{-\frac{2aq\Phi}{m}a^2} dp_y dp_z. \tag{11}$$

Then,

$$n(x) = \int_{a_1}^{a_2} da n_a(x). \tag{12}$$

The contribution of a particle species to the current density $J_y$ is

$$J_y(x) = q \int_{a_1}^{a_2} da \left( \frac{m}{2a_d q} \right) \frac{\partial n_a}{\partial A_y} = \int_{a_1}^{a_2} da T_a \frac{\partial n_a}{\partial A_y}, \tag{14}$$

$$J_z = \int_{a_1}^{a_2} da T_a \frac{\partial n_a}{\partial A_z}. \tag{15}$$

For a given species, the bulk velocity is simply its contribution to the current density divided by $q n$:

$$u_y = \frac{1}{f_{a_1}^{a_2} da n_a} \int_{a_1}^{a_2} da T_a \frac{\partial n_a}{q} \frac{\partial A_y}{\partial A_y} \tag{16}$$
$$u_z = \frac{1}{f_{a_1}^{a_2} da n_a} \int_{a_1}^{a_2} da T_a \frac{\partial n_a}{q} \frac{\partial A_z}{\partial A_z}.$$\tag{17}

The $u_x$ component, as $J_x$, is null.

4 The pressure tensor

By definition, the contribution of each species to the pressure tensor is

$$p = m \int (v - u)(v - u) f dv,$$ \tag{17}

where the tensor $(v - u)(v - u)$ is a dyadic product. Considering that the velocity $u_x$ is null,

$$p_{xx} = m \int v_x^2 f dv_x dp_y dp_z$$
$$= m \int \left\{ \left( \frac{a_d}{\pi} \right) \exp \left( -\frac{2aq\Phi}{m}a^2 \right) \right\} G_a(p_y, p_z) dp_y dp_z \times$$
$$\left\{ \frac{a_d}{\pi} \right\}^{1/2} v_x \left[ e^{-\alpha_a x^2} \right] da. \tag{18}$$
The terms between braces can be integrated separately. The first (integrated over \(dp, dp_z\)) is \(n_a\). The second (integrated over \(dv_z\)) is \((\frac{q}{m})^2\pi^{1/2}/2a^{3/2} = \frac{q}{m} n V_0\). In the end,

\[
p_{xx} = \int_{a_1}^{a_2} da \ n_a T_a . \tag{19}
\]

The computation of the other diagonal terms involves a finite bulk velocity,

\[
p_{zz} = \int_{a_1}^{a_2} da \ m \left(\frac{q}{m}\right) \int dp, dp_z \times \left\{ v_z^2 = 2v_z u_z + u_z^2 \right\} e^{-a[v_z^2+v_z^2]} e^{(-\frac{2aq}{m}G_a(p_y, p_z))} . \tag{20}
\]

The development of \((v_z - u_z)^2\) (in braces) can be cut into three parts, and \(p_{zz}\) is the sum of the three corresponding integrals. The second and the third integrals are simply \(-mvu_a^2\). The bulk velocity \(u_z\) can be eliminated using Eq. (16). In the first integral, \(v_z\) is eliminated with \(p_y\) and \(A_y\), and

\[
\left( \frac{q}{m} A_z \right)^2 e^{-a_0[(p_y - \frac{q}{m} A_y)^2 + (p_z - \frac{q}{m} A_z)^2]} = \left[ \left( \frac{m}{2a_0q} \right)^2 \partial^2 \frac{\partial^2}{\partial A_z^2} + \frac{1}{2a_0} \right] e^{-a_0[(p_y - \frac{q}{m} A_y)^2 + (p_z - \frac{q}{m} A_z)^2]} . \tag{21}
\]

Therefore, Eq. (20) can also be written

\[
p_{zz} = \int_{a_1}^{a_2} da \ T_a n_a + \frac{mT_a^2}{q^2} \partial^2 n_a \partial A_z^2 \right] - \frac{m}{\int da} n_a \left[ \int_{a_1}^{a_2} da \ T_a \left( \frac{\partial n_a}{\partial A_z} \right)^2 \right] . \tag{22}
\]

The relation for \(p_{yy}\) is analogous (some minus signs appear at different places but the final result is the same). The off-diagonal terms \(p_{xy}\) and \(p_{zx}\) are equal to zero, because they include a product by the integral over \(dv_z\) of an odd integrand. As

\[
\left( p_y - \frac{q}{m} A_y \right) \left( p_z - \frac{q}{m} A_z \right) e^{-a_0[(p_y - \frac{q}{m} A_y)^2 + (p_z - \frac{q}{m} A_z)^2]} = \left( \frac{m}{2a_0q} \right)^2 \partial^2 \frac{\partial^2}{\partial A_y \partial A_z} , e^{-a_0[(p_y - \frac{q}{m} A_y)^2 + (p_z - \frac{q}{m} A_z)^2]} , \tag{23}
\]

the off-diagonal term \(p_{yz}\) is

\[
p_{yz} = p_{zx} = \frac{mT_a^2}{q^2} \partial^2 n_a \partial A_y \partial A_z \right] - \frac{1}{\int da n_a^2} \int_{a_1}^{a_2} da T_a \frac{\partial n_a}{\partial A_y} \left( \int_{a_1}^{a_2} da T_a \frac{\partial n_a}{\partial A_z} \right) . \tag{24}
\]

Obviously, the pressure tensor contains four different terms. For the sake of simplicity, let us consider the case of the isothermal equilibria, set when \(T_0 = T\) is constant. Then the pressure terms simplify into

\[
p_{xx} = nT , \tag{25}
\]

\[
p_{yy} = nT \left[ 1 + \frac{mT_e}{q^2} \partial \frac{\partial}{\partial A_y} \left( \frac{1}{n \partial A_y} \right) \right] , \tag{26}
\]

\[
p_{zz} = nT \left[ 1 + \frac{mT_e}{q^2} \partial \frac{\partial}{\partial A_z} \left( \frac{1}{n \partial A_z} \right) \right] , \tag{27}
\]

\[
p_{yz} = nT^2 \partial \frac{\partial}{\partial A_y} \left( \frac{1}{n \partial A_z} \right) = nT^2 \partial \frac{\partial}{\partial A_y} \left( \frac{1}{n \partial A_z} \right) . \tag{28}
\]

If the magnetic field is everywhere parallel to \(z\), the off-diagonal terms vanish, \(p_{xx}\) and \(p_{yy}\) represent the pressure components in the directions perpendicular to the magnetic field, and \(p_{zz} = p_{jj}\) is the parallel component. The inequality \(p_{zz} \neq p_{jj}\) shows that the pressure tensor is not isotropic. Moreover, the perpendicular terms are different \(p_{xx} \neq p_{yy}\), therefore, the pressure tensor is non gyroscopic.

These non isotropic and non gyroscopic effects can be attributed to the finite Larmor radius \(\rho_L\). From a dimensional point of view, the terms \((p_{zz} - p_{xx})/(p_{xx})\) deduced from Eq. (26–27), scale as \((\frac{mT_e}{aT^2})^2 = (\rho_L k)^2\), where \(k\) is the inverse of the characteristic size of the density gradient. As long as \(\frac{\rho}{T} < \frac{m}{n},\) the non isotropic and non gyroscopic terms are predominantly carried by the ions.

### 5 First example: the Harris current sheet in the Earth’s magnetotail

In the simple example of the Harris current sheet, the pressure tensor can be expressed with elementary functions. This equilibrium corresponds to \(G_a(p_y, p_z) = n_0 + \Delta n \exp \nu p_y / m\) in Eq. (9), where \(\Delta n\) is the Dirac distribution. The density \(n_0\) is the density far from the discontinuity; it is arbitrary (it is null in the Harris paper (1962)). Defining \(\delta = v(z/m),\) the magnetic field and the contribution of each particle species to the density are

\[
B_z(x) = -B_0 \tanh \left( \frac{\delta B_0 x}{2} \right) \tag{29}
\]

\[
n(x) = n_0 + N_0 \exp \left( \delta A_y(x) \right) = n_0 + \frac{N_0}{\cosh \left( \frac{\delta B_0 x}{2} \right)^2} . \tag{30}
\]

The finite pressure components are \(p_{xx} = p_{yy} = nT\) and

\[
p_{yy} = nT \left[ 1 + \frac{mT^2}{q^2} \left( \frac{n - n_0}{} \right) \right] . \tag{31}
\]

In the case \(n_0 = 0\) of a bounded plasma (null density at infinity), the pressure tensor is a simple scalar tensor. It is a non gyroscopic tensor in all the other cases. Figure 1 shows an example of the Harris current sheet directly inspired from a tangential current sheet crossing studied by Petrukovich et al. (2003), and mentioned in the Introduction of this article. The parameters are inferred from the Cluster measurements: \(T_i = 1500\text{ eV}, B_0 = 20nT, n_0 = 1.5 \text{ cm}^{-3} .\)

We have set \(\delta = 1200 \text{ m}^{-1}\text{ T}^{-1}\), in order to set the layer thickness to 400 km (case of the 23:10 crossing (Petrukovich et al., 2003). We can see the magnetic field \(B_z\) reversal on a scale of 400 km. The pressure tensor is not gyroscopic.
ponents correspond to the gradient direction, and \( y \) in a low
Let us now briefly examine the case of a density structure
6 Second example: density cavities in the Earth’s auroral zone
with their ratio reaches \( p_{yy}/p_{xx}=p_{yy}/p_{zz}=4 \) in the middle
of the structure. The plasma is strongly non isotropic and non
gyrotrropic. In the case of a larger discontinuity, 1200 km,
(case of the 22:54 current sheet crossing), the ratio reaches a
smaller value: 1.3. Although less impressive than in the
previous case, the plasma pressure non gyrotrropy cannot be
neglected.

6 Second example: density cavities in the Earth’s auroral zone
Let us now briefly examine the case of a density structure
in a low \( \beta \) plasma. Such structures where shown by Mottez (2003)
to model the plasma cavities encountered in the high altitude Earth’s auroral zone. A simple case corresponds to \( G_\theta(p_y, p_z)=n_i \exp(-q/(p_y/m)^2) \). In such structures, the
size can reach the order of a few ion Larmor radii \( p_i \), but
cannot go below one ion Larmor radius. The magnetic field
remains quasi-uniform, in spite of large plasma density vari-
ations. With the (very accurate) approximation \( A_\parallel=B_\parallel x \), the
contribution of each particle species to the density is
\[
n(x)=n_0+N_0 e^{-\xi B_i^2 x^2}, \tag{32}
\]
where \( \xi \) is an (almost) arbitrary factor scaling the sharpness
of the density gradient. The pressure tensor is given by
\[
p_{yy}=nT \left[ 1 + \frac{-2mT N_0 \xi}{q^2} \right] \exp(-\xi B_i^2 x^2) \times
\left[ \frac{1-2\xi B_i^2 x^2}{n_0 + N_0 e^{-\xi B_i^2 x^2}} + \frac{2x^2 \xi N_0 e^{-\xi B_i^2 x^2}}{(n_0 + N_0 e^{-\xi B_i^2 x^2})^2} \right]. \tag{33}
\]
If \( n_0=0 \), \( p_{yy} \) becomes very simple. Let us define \( h \) by
\( \xi=q^2/(2mT h_i^2) \); it caracterizes the size of the structure com-
pared to the ion Larmor radius because (see Eq. (32))
n=\( N_0 \exp(x/h_i)^2 \). The \( p_{yy} \) pressure component is
\[
p_{yy}=nT \left[ 1-\frac{2mT \xi}{q^2} \right] = nT \left[ 1-\frac{1}{h_i^2} \right]. \tag{34}
\]
As long as the structure is large compared to the ion Larmor
radius (\( h_i \gg 1 \)), the pressure is nearly scalar. If the structure is
of the order of a few ion gyroradii (\( h_i \sim 1 \), the pressure
tensor becomes strongly non gyrotrropic, and \( p_{yy}<p_{xx}=p_{zz} \).
(The fact that \( p_{yy} \) becomes negative for \( h_i < 1 \) confirms that
the structure cannot be smaller than one ion gyroradius.)

7 Conclusion
The above computation of the pressure tensor associated with
tangential equilibria shows that we must be very carefull
when using a set of fluid equations to describe a tangen-
tial plasma structure: even in the simple cases given here
for illustration, the pressure tensor is non isotropic and non
gyrotrropic. Moreover, when the magnetic field direction
changes (a very common situation with tangential discontin-
uities), the off-diagonal terms cannot be neglected: this re-
inforces the non gyrotrropic character of the plasma pressure
tensor. Therefore, considering only the \( p_{yy} \) and \( p_{zz} \) com-
ponents of a diagonal pressure tensor cannot provide a good
description of the plasma. This is not a problem for a static
description, because only the \( p_{xx} \) component plays a role in
the fluid equations of the equilibrium. But when the stability
of the equilibrium or the magnetic reconnection are investi-
gated, for instance, through a perturbative analysis, the other
components of the equilibrium pressure tensor come into ac-
tion. Not considering them can be misleading.

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