The initial data problem for 3+1 numerical relativity
Part 1

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Plan

1. The initial data problem
2. Conformal transverse-traceless method
3. Conformal thin sandwich method
Outline

1. The initial data problem
2. Conformal transverse-traceless method
3. Conformal thin sandwich method
In lecture 1, we have seen

3+1 decomposition $\implies$ Einstein equation $= \text{Cauchy problem with constraints}$
Initial data for the Cauchy problem

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3+1 decomposition \(\implies\) Einstein equation = Cauchy problem with constraints

Constructing initial data: \(\exists\) two problems:

- **The mathematical problem:** given some hypersurface \(\Sigma_0\), find a Riemannian metric \(\gamma\), a symmetric bilinear form \(K\) and some matter distribution \((E, p)\) on \(\Sigma_0\) such that the Hamiltonian and momentum constraints are satisfied:

\[
R + K^2 - K_{ij}K^{ij} = 16\pi E
\]

\[
D_jK^j_i - D_iK = 8\pi p_i
\]

NB: the matter distribution \((E, p)\) may have some additional constraints from its own.
Initial data for the Cauchy problem

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\[ 3+1 \text{ decomposition} \implies \text{Einstein equation} = \text{Cauchy problem with constraints} \]

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D_jK^j_i - D_iK = 8\pi p_i
\]

NB: the matter distribution \( (E, p) \) may have some additional constraints from its own.

- **The astrophysical problem:** make sure that the obtained solution to the constraint equations have something to do with the physical system that one wish to study.
Notice that the constraints involve a single hypersurface $\Sigma_0$, not a foliation $(\Sigma_t)_{t \in \mathbb{R}}$. In particular, neither the lapse function $N$ nor the shift vector appear $\beta$ in these equations.
A first naive approach

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**Naive method of resolution:**

- choose freely the metric $\gamma$, thereby fixing the connection $D$ and the scalar curvature $R$
- solve the constraints for $K$

Indeed, for fixed $\gamma$, $E$, and $p$, the constraints form a quasi-linear system of first order for the components $K_{ij}$.
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Naive method of resolution:

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Indeed, for fixed $\gamma$, $E$, and $p$, the constraints form a quasi-linear system of first order for the components $K_{ij}$

However, this approach is not satisfactory:
only 4 equations for 6 unknowns $K_{ij}$ and there is no natural prescription for choosing arbitrarily two among the six components $K_{ij}$
Various approaches to the initial data problem

- **Conformal methods**: initiated by Lichnerowicz (1944) and extended by
  - Choquet-Bruhat (1956, 1971)
  - York and Ó Murchadha (1972, 1974, 1979)
→ by far the most widely spread techniques
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In this lecture we focus on **conformal methods**
Outline

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2. Conformal transverse-traceless method

3. Conformal thin sandwich method
Starting point

Conformal decomposition introduced in Lecture 1:

\[ \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad \text{and} \quad A^{ij} = \psi^{-10} \hat{A}^{ij} \]

The Hamiltonian and momentum constraints become respectively

\[ \tilde{D}_i \tilde{D}^i \psi - \frac{1}{8} \tilde{R} \psi + \frac{1}{8} \hat{A}_{ij} \hat{A}^{ij} \psi^{-7} + \left( 2\pi E - \frac{1}{12} K^2 \right) \psi^5 = 0 \]

\[ \tilde{D}_j \hat{A}^{ij} - \frac{2}{3} \psi^6 \tilde{D}^i K = 8\pi \psi^{10} p^i \]
Longitudinal/transverse decomposition of $\hat{A}^{ij}$

York (1973,1979) splitting of $\hat{A}^{ij}$:

$$\hat{A}^{ij} = (\tilde{L} X)^{ij} + \hat{A}^{ij}_{TT}$$

with

- $(\tilde{L} X)^{ij} = \text{conformal Killing operator}$ associated with the metric $\tilde{\gamma}$ and acting on the vector field $X$:

$$ (\tilde{L} X)^{ij} := \tilde{D}^i X^j + \tilde{D}^j X^i - \frac{2}{3} \tilde{D}_k X^k \tilde{\gamma}^{ij} $$

- $\hat{A}^{ij}_{TT}$ traceless and transverse (i.e. divergence-free) with respect to the metric $\tilde{\gamma}$: $\tilde{\gamma}^{ij} \hat{A}^{ij}_{TT} = 0$ and $\tilde{D}_j \hat{A}^{ij}_{TT} = 0$

NB: both the longitudinal part and the TT part are traceless: $\tilde{\gamma}^{ij} (\tilde{L} X)^{ij} = 0$ and $\tilde{\gamma}^{ij} \hat{A}^{ij}_{TT} = 0$
Determining $X$ and $\hat{A}^{ij}_{TT}$:

Considering the divergence of $\hat{A}^{ij}$, we see that $X$ must be a solution of the vector differential equation

$$\tilde{\Delta}_L X^i = \tilde{D}_j \hat{A}^{ij}$$

where $\tilde{\Delta}_L$ is the **conformal vector Laplacian**:

$$\tilde{\Delta}_L X^i := \tilde{D}_j (\tilde{L}X)^{ij} = \tilde{D}_j \tilde{D}^j X^i + \frac{1}{3} \tilde{D}^i \tilde{D}_j X^j + \tilde{R}^i_{\ j} X^j$$
Determining $X$ and $\hat{A}_{TT}^{ij}$:
Considering the divergence of $\hat{A}^{ij}$, we see that $X$ must be a solution of the vector differential equation

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where $\tilde{\Delta}_L$ is the conformal vector Laplacian:

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The operator $\tilde{\Delta}_L$ is elliptic and its kernel is reduced to conformal Killing vectors, i.e. vectors $C$ that satisfy $(\tilde{L}C)^{ij} = 0$ (generators of conformal isometries, if any)

- if $\Sigma_0$ is a closed manifold (i.e. compact without boundary): the solution $X$ exists; it may be not unique, but $(\tilde{L}X)^{ij}$ is unique;
- if $(\Sigma_0, \gamma)$ is an asymptotically flat manifold: there exists a unique solution $X$ which vanishes at spatial infinity

Conclusion: the longitudinal/transverse decomposition exists and is unique
Defining $\tilde{E} := \Psi^8 E$ and $\tilde{p}^i := \Psi^{10} p^i$, the Hamiltonian constraint (Lichnerowicz equation) and the momentum constraint become respectively

$$\tilde{D}_i \tilde{D}^i \Psi - \frac{\tilde{R}}{8} \Psi + \frac{1}{8} \left( (\tilde{L}X)_{ij} + \tilde{A}^T_{ij} \right) \left[ (\tilde{L}X)^{ij} + \tilde{A}^{ij}_T \right] \Psi^{-7} + 2\pi \tilde{E} \Psi^{-3} - \frac{K^2}{12} \Psi^5 = 0$$

(1)

$$\tilde{\Delta}_L X^i - \frac{2}{3} \Psi^6 \tilde{D}^i K = 8\pi \tilde{p}^i$$

(2)

where $(\tilde{L}X)_{ij} := \tilde{\gamma}_{ik} \tilde{\gamma}_{jl} (\tilde{L}X)^{kl}$ and $\tilde{A}^T_{ij} := \tilde{\gamma}_{ik} \tilde{\gamma}_{jl} \tilde{A}^{kl}_T$
In view of the above system, we see clearly which part of the initial data on $\Sigma_0$ can be freely chosen and which part is “constrained”:

- **free data:**
  - conformal metric $\tilde{\gamma}$
  - symmetric traceless and transverse\(^1\) tensor $\hat{A}^{TT}_{ij}$
  - scalar field $K$
  - conformal matter variables: $(\tilde{E}, \tilde{p}^i)$

- **constrained data** (or “determined data”):
  - conformal factor $\Psi$, obeying the *non-linear* elliptic equation (1)
  - vector $X$, obeying the *linear* elliptic equation (2)

\(^1\)traceless and transverse are meant with respect to $\tilde{\gamma}$
York (1979) CTT method:

1. choose \((\tilde{\gamma}_{ij}, \hat{A}_{ij}^{TT}, K, \tilde{E}, \tilde{p}^i)\) on \(\Sigma_0\)
2. solve the system (1)-(2) to get \(\Psi\) and \(X^i\)
3. construct

\[
\begin{align*}
\gamma_{ij} &= \Psi^4 \tilde{\gamma}_{ij} \\
K^{ij} &= \Psi^{-10} \left( (\tilde{L}X)^{ij} + \hat{A}_{ij}^{TT} \right) + \frac{1}{3} \Psi^{-4} K \tilde{\gamma}^{ij} \\
E &= \Psi^{-8} \tilde{E} \\
p^i &= \Psi^{-10} \tilde{p}^i
\end{align*}
\]

Then one obtains a set \((\gamma, K, E, p)\) which satisfies the constraint equations
Consider the momentum constraint equation:
\[ \tilde{\Delta}_L X^i - \frac{2}{3} \psi^6 \tilde{D}^i K = 8\pi \tilde{p}^i \]

If \( \Sigma_0 \) has a \textbf{constant mean curvature (CMC)}:
\[ K = \text{const} \]

then \( \tilde{D}^i K = 0 \) and the momentum constraint equations reduces to
\[ \tilde{\Delta}_L X^i = 8\pi \tilde{p}^i \] (3)

It does no longer involve \( \psi \)
\[ \longrightarrow \text{decoupling of the constraint system (1)-(2)} \]

\textit{NB:} a very important case of CMC hypersurface: \textbf{maximal hypersurface}: \( K = 0 \)
1st step: Solve the linear elliptic equation (3) \( \tilde{\Delta}_L X^i = 8\pi \tilde{\rho}^i \) to get the vector \( X \)
- if \( \Sigma_0 \) is a closed manifold (i.e. compact without boundary): the solution \( X \) exists; it may be not unique, but \( (\tilde{L}X)^{ij} \) is unique;
- if \( (\Sigma_0, \gamma) \) is an asymptotically flat manifold: there exists a unique solution \( X \) which vanishes at spatial infinity

2nd step: Inject the solution \( X \) into Lichnerowicz equation (1)

\[
\tilde{D}_i \tilde{D}^i \Psi - \frac{\tilde{R}}{8} \Psi + \frac{1}{8\Psi^7} \left[ (\tilde{L}X)^{ij} + \hat{A}_{ij}^{TT} \right] \left[ (\tilde{L}X)^{ij} + \hat{A}_{ij}^{TT} \right] + \frac{2\pi \tilde{E}}{\Psi^3} - \frac{K^2}{12} \Psi^5 = 0
\]

and solve the latter for \( \Psi \) (the difficult part !)
Conformal transverse-traceless method

Strategy on CMC hypersurfaces

1st step: Solve the linear elliptic equation (3) \( \tilde{\Delta}_L X^i = 8\pi \tilde{p}^i \) to get the vector \( X \)

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Existence and uniqueness of solutions to Lichnerowicz equation:

- asymptotically flat case: (1) is solvable iff the metric \( \tilde{\gamma} \) is conformal to a metric with vanishing scalar curvature (Cantor 1977)
- closed manifold: complete analysis carried out by Isenberg (1995) (vacuum case)

More details: see review by Bartnik and Isenberg (2004)
Conformally flat initial data on maximal slices

Simplest choice for free data \((\tilde{\gamma}_{ij}, \hat{A}^{TT}_{ij}, K, \tilde{E}, \tilde{p}^i)\):

- \(\tilde{\gamma}_{ij} = f_{ij}\) (flat metric)
- \(\hat{A}^{TT}_{ij} = 0\)
- \(K = 0\) (\(\Sigma_0 = \) maximal hypersurface)
- \(\tilde{E} = 0\) and \(\tilde{p}^i = 0\) (vacuum)
Conformal transverse-traceless method

Conformally flat initial data on maximal slices

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- \(K = 0\) \((\Sigma_0 = \text{maximal hypersurface})\)
- \(\tilde{E} = 0\) and \(\tilde{p}^i = 0\) (vacuum)

Then the constraint equations (1)-((2) reduce to

\[
\Delta \psi + \frac{1}{8} (LX)_{ij} (LX)^{ij} \psi^{-7} = 0 \tag{4}
\]

\[
\Delta X^i + \frac{1}{3} \mathcal{D}^i \mathcal{D}_j X^j = 0 \tag{5}
\]

where \(\Delta := \mathcal{D}_i \mathcal{D}^i\) (flat Laplacian) and \((LX)^{ij} := \mathcal{D}^i X^j + \mathcal{D}^j X^i - \frac{2}{3} \mathcal{D}_k X^k f^{ij}\)

\((\mathcal{D}_i\ \text{flat connection: in Cartesian coordinates } \mathcal{D}_i = \partial_i)\)

Asymptotic flatness \(\Rightarrow\) boundary conditions \[
\begin{align*}
\left. \psi \right|_{r \to \infty} &= 1 \\
\left. X \right|_{r \to \infty} &= 0
\end{align*}
\]
Choose $\Sigma_0 \sim \mathbb{R}^3$

Then the only regular solution to $\Delta X^i + \frac{1}{3} D^i D_j X^j = 0$ with the boundary condition $X|_{r \to \infty} = 0$ is

$$X = 0$$
A (too) simple solution

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Plugging this solution into the Hamiltonian constraint (4) yields Laplace equation for $\Psi$:

$$\Delta \Psi = 0$$

With the boundary condition $\Psi|_{r \to \infty} = 1$ the unique regular solution is

$$\Psi = 1$$

Hence the initial $(\gamma, K)$ is \[\gamma_{ij} = f_{ij}\]

$$K_{ij} = 0 \quad \text{(momentarily static)}$$

This is a standard slice $t = \text{const}$ of Minkowski spacetime
A less trivial solution

Keep the same simple free data as above, but choose for $\Sigma_0$ a less trivial topology: $\Sigma_0 \sim \mathbb{R}^3 \setminus B$ ($B=$ball):

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Boundary conditions (BC) for $X$ and $\Psi$ must be supplied at the sphere $S$ delimiting $B$

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$$\Sigma_0 \sim \mathbb{R}^3 \setminus \mathcal{B} \quad (\mathcal{B} =$ ball).$$

boundary conditions (BC) for $X$ and $\Psi$ must be supplied at the sphere $S$ delimiting $\mathcal{B}$

Let us choose $X|_S = 0$. Altogether with the outer BC $X|_{r \to \infty} = 0$ this yields to the following solution of momentum constraint (5)

$$X = 0$$
A less trivial solution

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\[ \Sigma_0 \]

\[ \mathbb{R}^3 \setminus \mathcal{B} \]

\[ \mathcal{B} \]

\[ \mathcal{B} = \text{ball} \]

\[ \Rightarrow \]

boundary conditions (BC) for $X$ and $\Psi$ must be supplied at the sphere $S$ delimiting $\mathcal{B}$

Let us choose $X|_S = 0$. Altogether with the outer BC $X|_{r \to \infty} = 0$ this yields to the following solution of momentum constraint (5)

\[ X = 0 \]

Hamiltonian constraint (4) $\Rightarrow$ Laplace equation $\Delta \Psi = 0$

The choice $\Psi|_S = 1$ would result in the same trivial solution $\Psi = 1$ as before...
A less trivial solution

In order to have something not trivial, i.e. to ensure that the metric $\gamma$ will not be flat, let us demand that $\gamma$ admits a **closed minimal surface**:

$s$ : unit normal to $S$ for the metric $\gamma$

$\tilde{s}$ : unit normal to $S$ for the metric $\tilde{\gamma}$

\[ s \text{ for } \gamma \quad \tilde{s} \text{ for } \tilde{\gamma} \]

\[ (r, \theta, \varphi) : \text{coord. sys.} / f_{ij} = \text{diag}(1, r^2, r^2 \sin^2 \theta) \text{ and } S = \text{sphere } \{r = a\} \]

\[ \boxed{\left( \frac{\partial \Psi}{\partial r} + \frac{\Psi}{2r} \right)_{r=a} = 0 } \]

$S$ minimal surface

\[ \iff S' \text{’s mean curvature } = 0 \]

\[ \iff D_i s^i \big|_S = 0 \]

\[ \iff D_i (\Psi^6 s^i) \big|_S = 0 \]

\[ \iff D_i (\Psi^4 \tilde{s}^i) \big|_S = 0 \]
A less trivial solution

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\[ S \text{ minimal surface} \]

\[ \iff \text{S's mean curvature} = 0 \]

\[ \iff D_i s^i |_S = 0 \]

\[ \iff D_i (\psi^6 s^i)|_S = 0 \]

\[ \iff D_i (\psi^4 \tilde{s}^i)|_S = 0 \]

\[ \iff \frac{\partial \psi}{\partial r} + \frac{\psi}{2r} \bigg|_{r=a} = 0 \quad (6) \]

$s$: unit normal to $S$ for the metric $\gamma$

$\tilde{s}$: unit normal to $S$ for the metric $\tilde{\gamma}$

$(r, \theta, \varphi)$: coord. sys. / $f_{ij} = \text{diag}(1, r^2, r^2 \sin^2 \theta)$ and $S = \text{sphere} \{ r = a \}$

The solution to Laplace equation $\Delta \psi = 0$ with the BC (6) and $\psi|_{r \to \infty} = 1$ is

\[ \psi = 1 + \frac{a}{r} \]
Conformal transverse-traceless method

A less trivial solution

\[ m = -\frac{1}{2\pi} \lim_{r \to \infty} \oint_{r=\text{const}} \partial_{\Psi} r^2 \sin \theta \, d\theta \, d\varphi \]

\[ \Rightarrow m = 2a \]

Hence \[ \Psi = 1 + \frac{m}{2r} \]

The obtained initial data is then

\[ \begin{cases} 
\gamma_{ij} = \left(1 + \frac{m}{2r}\right)^4 \text{diag}(1, r^2, r^2 \sin \theta) \\
K_{ij} = 0
\end{cases} \]
A less trivial solution

ADM mass of that solution:

\[ m = -\frac{1}{2\pi} \lim_{r \to \infty} \int_{r=\text{const}} \partial\psi \frac{r^2 \sin \theta \, d\theta \, d\varphi}{r} \]

\[ \Rightarrow m = 2a \]

Hence \[ \Psi = 1 + \frac{m}{2r} \]

The obtained initial data is then

\[ \left\{ \begin{array}{l}
\gamma_{ij} = (1 + \frac{m}{2r})^4 \text{diag}(1, r^2, r^2 \sin \theta) \\
K_{ij} = 0
\end{array} \right. \]

This is a slice \( t = \text{const} \) of Schwarzschild spacetime

Remember: Schwarzschild metric in isotropic coordinates \((t, r, \theta, \varphi)\):

\[ g_{\mu\nu} dx^\mu dx^\nu = -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2 dt^2 + \left(1 + \frac{m}{2r}\right)^4 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] \]

Link with Schwarzschild coordinates \((t, R, \theta, \varphi)\): \[ R = r \left(1 + \frac{m}{2r}\right)^2 \]
Extended solution

$S$ minimal surface $\implies (\Sigma_0, \gamma)$ can be extended *smoothly* to a larger Riemannian manifold $(\Sigma'_0, \gamma')$ by gluing a copy of $\Sigma_0$ at $S$:
$S$ minimal surface $\implies (\Sigma_0, \gamma)$ can be extended \textit{smoothly} to a larger Riemannian manifold $(\Sigma'_0, \gamma')$ by gluing a copy of $\Sigma_0$ at $S$:

$S = \text{Einstein-Rosen bridge}$

between two asymptotically flat manifolds

range of $r$ in $\Sigma'_0$: $(0, +\infty)$

extended metric :

$\gamma'_{ij} \, dx^i \, dx^j = \left(1 + \frac{m}{2r}\right)^4 \times$

$(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)$

region $r \to 0 = \text{second asymptotically flat region}$

map $r \mapsto r' = \frac{m^2}{4r}$ is an isometry

This extended solution is still a slice $t = \text{const}$ of Schwarzschild spacetime

topology of $\Sigma'_0 = \mathbb{R}^3\setminus\{O\}$ (puncture)
The Bowen-York solution

Same free data as before:
\[ \tilde{\gamma}_{ij} = f_{ij}, \hat{A}^{TT}_{ij} = 0, K = 0, \tilde{E} = 0 \text{ and } \hat{p}^i = 0 \]
so that the constraint equations are still

\[ \Delta \psi + \frac{1}{8} (LX)_{ij} (LX)^{ij} \psi^{-7} = 0 \]  
(7)

\[ \Delta X^i + \frac{1}{3} D^i D_j X^j = 0 \]  
(8)

Choice of \( \Sigma_0 \): \( \Sigma_0 = \mathbb{R}^3 \setminus \{O\} \) (puncture topology)
Conformal transverse-traceless method

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\[ \Delta \psi + \frac{1}{8} (LX)_{ij} (LX)^{ij} \psi^{-7} = 0 \quad (7) \]
\[ \Delta X^i + \frac{1}{3} \mathcal{D}^i \mathcal{D} j X^j = 0 \quad (8) \]

Choice of \( \Sigma_0 \): \( \Sigma_0 = \mathbb{R}^3 \setminus \{O\} \) (puncture topology)
Difference with previous case: \( X \neq 0 \) (no longer momentarily static data)

Bowen-York (1980) solution of Eq. (8) in Cartesian coord. \( (x^i) = (x, y, z) \):

\[ X^i = -\frac{1}{4r} \left( 7P^i + P_j \frac{x^j x^i}{r^2} \right) - \frac{1}{r^3} \epsilon^{i}_{jk} S^j x^k \]

Two constant vector parameters : \( \begin{cases} P^i = \text{ADM linear momentum} \\ S^i = \text{angular momentum} \end{cases} \)
Example: choose $S^i$ perpendicular to $P^i$ and choose Cartesian coordinates $(x, y, z)$ such that $P^i = (0, P, 0)$ and $S^i = (0, 0, S)$. Then

$$X^x = -\frac{P}{4} \frac{xy}{r^3} + \frac{S}{r^3} \frac{y}{r^3}$$

$$X^y = -\frac{P}{4r} \left(7 + \frac{y^2}{r^2}\right) - \frac{S}{r^3} \frac{x}{r^3}$$

$$X^z = -\frac{P}{4} \frac{xz}{r^3}$$

Bowen-York extrinsic curvature: $\hat{A}^{ij} = (LX)^{ij}$:

$$\hat{A}^{ij} = \frac{3}{2r^3} \left[P^i x^j + P^j x^i - \left(\delta^{ij} - \frac{x^i x^j}{r^2}\right) P^k x_k\right] + \frac{3}{r^5} \left(\epsilon^{ikl} S^k x^l x^j + \epsilon^{jkl} S^k x^l x^i\right)$$

ADM linear momentum: $P_i := \frac{1}{8\pi} \lim_{S_t \to \infty} \int_{S_t} (K_{jk} - K_{\gamma_{jk}}) (\partial_i)^j s^k \sqrt{q} \, d^2 y$

Angular momentum (QI): $S_i := \frac{1}{8\pi} \lim_{S_t \to \infty} \int_{S_t} (K_{jk} - K_{\gamma_{jk}}) (\phi_i)^j s^k \sqrt{q} \, d^2 y$. 
There remains to solve (numerically !) the Hamiltonian constraint equation (7):

\[ \Delta \psi + \frac{1}{8} \hat{A}_{ij} \hat{A}^{ij} \psi^{-7} = 0 \]

and to reconstruct

\[
\begin{aligned}
\gamma_{ij} &= \psi^4 f_{ij} \\
K_{ij} &= \psi^{-2} \hat{A}_{ij}
\end{aligned}
\]
The Bowen-York solution

There remains to solve (numerically !) the Hamiltonian constraint equation (7):

$$\Delta \psi + \frac{1}{8} \hat{A}_{ij} \hat{A}^{ij} \psi^{-7} = 0$$

and to reconstruct

$$\begin{cases} 
    \gamma_{ij} = \psi^4 f_{ij} \\
    K_{ij} = \psi^{-2} \hat{A}_{ij}
\end{cases}$$

Remark 1: static Bowen-York solution ($P^i = 0, S^i = 0$) = maximal slice of Schwarzschild spacetime considered above

Remark 2: Bowen-York solution with $S^i \neq 0$ is not a slice of Kerr spacetime : it is initial data for a rotating black hole but in a non stationary state (black hole “surrounded” by gravitational radiation)
Outline

1. The initial data problem
2. Conformal transverse-traceless method
3. Conformal thin sandwich method
Conformal thin sandwich decomposition of extrinsic curvature


From Lecture 1: 
\[
\left( \frac{\partial}{\partial t} - L_\beta \right) \tilde{\gamma}^{ij} = 2N \tilde{A}^{ij} + \frac{2}{3} \tilde{D}_k \beta^k \tilde{\gamma}^{ij}
\]

with \( \tilde{A}^{ij} = \psi^4 A^{ij} = \psi^{-6} \hat{A}^{ij} \) and 
\[-L_\beta \tilde{\gamma}^{ij} = (\tilde{L} \beta)^{ij} + \frac{2}{3} \tilde{D}_k \beta^k \]

Hence
\[
\hat{A}^{ij} = \frac{\psi^6}{2N} \left[ \hat{\gamma}^{ij} + (\tilde{L} \beta)^{ij} \right]
\]

where \( \hat{\gamma}^{ij} := \frac{\partial}{\partial t} \tilde{\gamma}^{ij} \)

Introduce the **conformal lapse**: \( \tilde{N} := \psi^{-6} N \)

then
\[
\hat{A}^{ij} = \frac{1}{2\tilde{N}} \left[ \hat{\gamma}^{ij} + (\tilde{L} \beta)^{ij} \right]
\]
Hamiltonian and momentum constraints become

\[ \tilde{D}_i \tilde{D}^i \Psi - \frac{\tilde{R}}{8} \Psi + \frac{1}{8} \tilde{A}_{ij} \tilde{A}^{ij} \Psi^{-7} + 2\pi \tilde{E} \Psi^{-3} - \frac{K^2}{12} \Psi^5 = 0 \]

\[ \tilde{D}_j \left( \frac{1}{\tilde{N}} (\tilde{L} \beta)^{ij} \right) + \tilde{D}_j \left( \frac{1}{\tilde{N}} \tilde{\gamma}^{ij} \right) - \frac{4}{3} \Psi^6 \tilde{D}^i K = 16\pi \tilde{p}^i \]

- **free data**: \((\tilde{\gamma}_{ij}, \tilde{\gamma}^{ij}, K, \tilde{N}, \tilde{E}, \tilde{p}^i)\)
- **constrained data**: \(\Psi\) and \(\beta^i\)
Extended conformal thin sandwich (XCTS)

**Origin:** Pfeiffer & York (2003)

**Idea:** instead of choosing the conformal lapse \( \tilde{N} \), compute it from the Einstein equation (not a constraint!) involving the time derivative \( \dot{K} \) of \( K \):

from Lecture 1:

\[
\left( \frac{\partial}{\partial t} - \mathcal{L}_\beta \right) K = -\psi^{-4} (\tilde{D}_i \tilde{D}^i N + 2\tilde{D}_i \ln \psi \tilde{D}^i N) \\
+ N \left[ 4\pi (E + S) + \tilde{A}_{ij} \tilde{A}^{ij} + \frac{K^2}{3} \right]
\]

Combining with the Hamiltonian constraint, we get

\[
\tilde{D}_i \tilde{D}^i (\tilde{N} \psi^7) - (\tilde{N} \psi^7) \left[ \frac{1}{8} \tilde{R} + \frac{5}{12} K^2 \psi^4 + \frac{7}{8} \tilde{A}_{ij} \tilde{A}^{ij} \psi^{-8} + 2\pi (\tilde{E} + 2\tilde{S}) \psi^{-4} \right] \\
+ \left( \dot{K} - \beta^i \tilde{D}_i K \right) \psi^5 = 0
\]

where \( \tilde{E} := \psi^8 E \) and \( \tilde{S} := \psi^8 S \)
PDE system of 5 equations:

\[
\begin{align*}
\tilde{D}_i \tilde{D}^i \psi - \frac{\tilde{R}}{8} \psi + \frac{1}{8} \hat{A}_{ij} \hat{A}^{ij} \psi^{-7} + 2\pi \tilde{E} \psi^{-3} - \frac{K^2}{12} \psi^5 &= 0 \\
\tilde{D}_j \left( \frac{1}{\tilde{N}} (\tilde{L} \beta)^{ij} \right) + \tilde{D}_j \left( \frac{1}{\tilde{N}} \hat{\gamma}^{ij} \right) - \frac{4}{3} \psi^6 \tilde{D}^i K - 16\pi \tilde{p}^i &= 0 \\
\tilde{D}_i \tilde{D}^i (\tilde{N} \psi^7) - (\tilde{N} \psi^7) \left[ \frac{1}{8} \tilde{R} + \frac{5}{12} K^2 \psi^4 + \frac{7}{8} \hat{A}_{ij} \hat{A}^{ij} \psi^{-8} + 2\pi (\tilde{E} + 2\tilde{S}) \psi^{-4} \right] \\
&\quad + \left( \dot{K} - \beta^i \tilde{D}_i K \right) \psi^5 = 0
\end{align*}
\]

- **Free data**: \((\tilde{\gamma}_{ij}, \hat{\gamma}^{ij}, K, \dot{K}, \tilde{E}, \tilde{S}, \tilde{p}^i)\)
- **Constrained data**: \(\psi, \tilde{N}\) and \(\beta^i\)
Pfeiffer & York (2005): in some cases, solutions \((\psi, \tilde{N}, \beta^i)\) to the (non-linear !) XCTS system are not unique, even on maximal surfaces

See also analysis by Baumgarte, Ó Murchadha & Pfeiffer (2007) and Walsh (2007)
XCTS at work: a simple example

Choose the same manifold $\Sigma_0 = \mathbb{R}^3 \setminus B$ ($\mathbb{R}^3$ with an excised ball) as considered previously.

Choose the free data to be

\[ \tilde{\gamma}_{ij} = f_{ij}, \tilde{\gamma}^{ij} = 0, K = 0, \dot{K} = 0, \tilde{E} = 0, \tilde{\mathcal{S}} = 0, \tilde{\rho}^i = 0 \]
Choose the same manifold $\Sigma_0 = \mathbb{R}^3 \setminus B$ ($\mathbb{R}^3$ with an excised ball) as considered previously.

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$\Rightarrow$ the XCTS equations reduce to

\[
\Delta \psi + \frac{1}{8} \hat{A}_{ij} \hat{A}^{ij} \psi^{-7} = 0 \tag{9}
\]

\[
\mathcal{D}_j \left( \frac{1}{\tilde{N}} (L\beta)^{ij} \right) = 0 \tag{10}
\]

\[
\Delta (\tilde{N} \psi^7) - \frac{7}{8} \hat{A}_{ij} \hat{A}^{ij} \psi^{-1} \tilde{N} = 0 \tag{11}
\]

with $\hat{A}_{ij} = \frac{1}{2\tilde{N}} (L\beta)^{ij}$
Conformal thin sandwich method

XCTS at work: a simple example

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Choose the free data to be
\[
\tilde{\gamma}_{ij} = f_{ij}, \quad \dot{\tilde{\gamma}}_{ij} = 0, \quad K = 0, \quad \dot{K} = 0, \quad \tilde{E} = 0, \\
\tilde{S} = 0, \quad \tilde{p}^i = 0
\]

\[\Rightarrow\] the XCTS equations reduce to

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with \( \hat{A}^{ij} = \frac{1}{2\tilde{N}} (L\beta)^{ij} \)

Choose the boundary condition \( \beta|_S = 0 \) in addition to \( \beta|_{r \to \infty} = 0 \). Then, independently of the value of \( \tilde{N} \), the unique solution to Eq. (10) is

\[\beta = 0\]
XCTS at work: a simple example

Accordingly \( \hat{A}^{ij} = 0 \) and Eqs. (9) and (11) reduce to two Laplace equations:

\[
\begin{align*}
\Delta \psi &= 0 \\
\Delta (\tilde{N} \psi^7) &= 0
\end{align*}
\]

As previously use the minimal surface requirement for \( S \) to get the solution

\[
\psi = 1 + \frac{m}{2r}
\]

to Eq. (12).
Conformal thin sandwich method

XCTS at work: a simple example

Accordingly $\hat{A}^{ij} = 0$ and Eqs. (9) and (11) reduce to two Laplace equations:

$$\Delta \psi = 0$$  \hspace{1cm} (12)

$$\Delta (\tilde{N} \psi^7) = 0$$  \hspace{1cm} (13)

As previously use the minimal surface requirement for $S$ to get the solution $\psi = 1 + \frac{m}{2r}$ to Eq. (12).

Regarding Eq. (13), choose the BC $\tilde{N}|_S = 0$ (singular slicing). Along with the asymptotic flatness BCs $\tilde{N}|_{r\to\infty} = 1$ and $\psi|_{r\to\infty} = 1$, this yields the solution

$$\tilde{N} \psi^7 = 1 - \frac{m}{2r}, \text{ i.e., since } N = \psi^6 \tilde{N}, \quad N = \left(1 - \frac{m}{2r}\right) \left(1 + \frac{m}{2r}\right)^{-1}$$
XCTS at work: a simple example

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$$\Delta \psi = 0$$
$$\Delta (\tilde{N} \psi^7) = 0$$

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$$\tilde{N} \psi^7 = 1 - \frac{m}{2r}, \text{ i.e., since } N = \psi^6 \tilde{N}, \quad N = \left(1 - \frac{m}{2r}\right) \left(1 + \frac{m}{2r}\right)^{-1}$$

We obtain Schwarzschild metric (in isotropic coordinates):

$$g_{\mu\nu} dx^\mu dx^\nu = - \left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2 dt^2 + \left(1 + \frac{m}{2r}\right)^4 \left[dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)\right]$$
Comparing CTT and (X)CTS methods

- CTT: choose some transverse traceless part $\hat{A}_{TT}^{ij}$ of the extrinsic curvature $K^{ij}$, i.e., some momentum $^2 \Rightarrow \text{CTT} = \text{Hamiltonian representation}$

- CTS or XCTS: choose some time derivative $\dot{\tilde{\gamma}}^{ij}$ of the conformal metric $\tilde{\gamma}^{ij}$, i.e., some velocity $\Rightarrow (X)\text{CTS} = \text{Lagrangian representation}$

2\text{recall the relation } \pi^{ij} = \sqrt{\gamma}(K\gamma^{ij} - K^{ij}) \text{ between } K^{ij} \text{ and the ADM canonical momentum}
Comparing CTT and (X)CTS methods

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Advantage of CTT: mathematical theory well developed; existence and uniqueness of solutions established (at least for constant mean curvature ($K = \text{const}$) slices)

Advantage of XCTS: better suited to the description of quasi-stationary spacetimes ($\rightarrow$ quasiequilibrium initial data): 
\[
\frac{\partial}{\partial t} \text{ Killing vector } \Rightarrow \ddot{\tilde{\gamma}}^{ij} = 0 \text{ and } \dot{K} = 0
\]

\[^2\text{recall the relation } \pi^{ij} = \sqrt{\gamma}(K \gamma^{ij} - K^{ij}) \text{ between } K^{ij} \text{ and the ADM canonical momentum} \]