Modelling black holes as trapping horizons

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Plan

1. Local approaches to black holes

2. Viscous fluid analogy

3. Angular momentum and area evolution laws

4. Applications to numerical relativity
Outline

1 Local approaches to black holes

2 Viscous fluid analogy

3 Angular momentum and area evolution laws

4 Applications to numerical relativity
Classical definition of a black hole

black hole: $\mathcal{B} := \mathcal{M} - J^-(\mathcal{I}^+)$

i.e. the region of spacetime where light rays cannot escape to infinity

- $(\mathcal{M}, g) =$ asymptotically flat manifold
- $\mathcal{I}^+ =$ future null infinity
- $J^-(\mathcal{I}^+) =$ causal past of $\mathcal{I}^+$
Local approaches to black holes

Classical definition of a black hole

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- $(\mathcal{M}, g) = \text{asymptotically flat manifold}$
- $\mathcal{I}^+ = \text{future null infinity}$
- $J^-(\mathcal{I}^+) = \text{causal past of } \mathcal{I}^+$

event horizon: $\mathcal{H} := \partial J^-(\mathcal{I}^+)$

(boundary of $J^-(\mathcal{I}^+)$)

$\mathcal{H}$ smooth $\implies \mathcal{H}$ null hypersurface
Drawbacks of the classical definition

- not applicable in **cosmology**, for in general $(\mathcal{M}, g)$ is not asymptotically flat
Local approaches to black holes

**Drawbacks of the classical definition**

- not applicable in *cosmology*, for in general \((\mathcal{M}, g)\) is not asymptotically flat
- even when applicable, this definition is highly non-local:

\[
ds^2 = -(1 - 2m(v)r) dv^2 + 2 dv dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

with \(m(v) = 0\) for \(v < 0\), \(dm/dv > 0\) for \(0 \leq v \leq v_0\), \(m(v) = M_0\) for \(v > v_0\)

⇒ no local physical experiment whatsoever can locate the event horizon
Determination of $\dot{J}^-(\mathcal{I}^+)$ requires the knowledge of the entire future null infinity. Moreover this is not locally linked with the notion of strong gravitational field:

**Example of event horizon in a flat region of spacetime:**

Vaidya metric, describing incoming radiation from infinity:

$$ds^2 = - \left( 1 - \frac{2m(v)}{r} \right) dv^2 + 2dv dr + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$

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[Ashtekar & Krishnan, LRR 7, 10 (2004)]
Local approaches to black holes

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\[\Rightarrow\] no local physical experiment whatsoever can locate the event horizon

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Another non-local feature: teleological nature of event horizons

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To deal with black holes as ordinary physical objects, a local definition would be desirable
→ quantum gravity, numerical relativity

Recently a **new paradigm** appeared in the theoretical approach of black holes: instead of *event horizons*, black holes are described by

- trapping horizons (Hayward 1994)
- isolated horizons (Ashtekar et al. 1999)
- dynamical horizons (Ashtekar and Krishnan 2002)
- slowly evolving horizons (Booth and Fairhurst 2004)

All these concepts are **local** and are based on the notion of **trapped surfaces**
Definition of a trapped surface

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Being spacelike, \( S \) lies outside the light cone
∃ two future-directed null directions orthogonal to \( S \):
\( \ell = \) outgoing, expansion \( \theta(\ell) \)
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$S$ is trapped $\iff \theta(k) < 0$ and $\theta(\ell) < 0$  
$S$ is marginally trapped $\iff \theta(k) < 0$ and $\theta(\ell) = 0$  

[Penrose 1965]
Local approaches to black holes

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trapped surface = local concept characterizing very strong gravitational fields
Local definitions of “black holes”

A hypersurface $\mathcal{H}$ of $(\mathcal{M}, g)$ is said to be

- a **future outer trapping horizon (FOTH)** iff
  (i) $\mathcal{H}$ foliated by marginally trapped 2-surfaces
  $(\theta^{(k)} < 0$ and $\theta^{(\ell)} = 0$)
  (ii) $\mathcal{L}_k \theta^{(\ell)} < 0$ (locally outermost trapped surf.)

[Hayward, PRD 49, 6467 (1994)]
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- **an isolated horizon (IH)** iff
  1. $\mathcal{H}$ is a non-expanding horizon
  2. $\mathcal{H}$’s full geometry is not evolving along the null generators: $[\mathcal{L}_\ell, \hat{\nabla}] = 0$
  [Ashtekar, Beetle & Fairhurst, CQG 16, L1 (1999)]
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The *trapping horizons* and *dynamical horizons* have their **own dynamics**, ruled by Einstein equations. In particular, one can establish for them

- existence and (partial) uniqueness theorems
  
  [Andersson, Mars & Simon, PRL 95, 111102 (2005)],

- first and second laws of black hole mechanics
  
  [Ashtekar & Krishnan, PRD 68, 104030 (2003)], [Hayward, PRD 70, 104027 (2004)]

- a viscous fluid bubble analogy (“membrane paradigm”, as for the event horizon)
  
  [EG, PRD 72, 104007 (2005)], [EG & Jaramillo, PRD 74, 087502 (2006)]

Foliation of a hypersurface by spacelike 2-surfaces

A hypersurface $\mathcal{H}$ is a submanifold of spacetime $(\mathcal{M}, g)$ of codimension 1. $\mathcal{H}$ can be spacelike, null, or timelike.

$$\mathcal{H} = \bigcup_{t \in \mathbb{R}} S_t$$

$S_t$ is a spacelike 2-surface.
Local approaches to black holes

Foliation of a hypersurface by spacelike 2-surfaces

hypersurface $\mathcal{H} =$ submanifold of spacetime $(\mathcal{M}, g)$ of codimension 1

$\mathcal{H}$ can be

- spacelike
- null
- timelike

$\mathcal{H} = \bigcup_{t \in \mathbb{R}} S_t$

$S_t = \text{spacelike 2-surface}$

$\sum_{t+dt} \rightarrow 3+1 \text{ perspective}$
Foliation of a hypersurface by spacelike 2-surfaces

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$\mathcal{H}$ can be

$\begin{cases} 
\text{spacelike} \\
\text{null} \\
\text{timelike}
\end{cases}$

$\mathcal{H} = \bigcup_{t \in \mathbb{R}} S_t$

$S_t = $ spacelike 2-surface

intrinsic viewpoint adopted here (i.e. not relying on extra-structure such as a 3+1 foliation)

$q :$ induced metric on $S_t$ (positive definite)

$\mathcal{D} :$ connection associated with $q$
Local approaches to black holes

Evolution vector on the horizon

Vector field $h$ on $\mathcal{H}$ defined by

1. $h$ is tangent to $\mathcal{H}$
2. $h$ is orthogonal to $S_t$
3. $\mathcal{L}_h t = h^\mu \partial_\mu t = \langle dt, h \rangle = 1$

NB: (iii) $\implies$ the 2-surfaces $S_t$ are Lie-dragged by $h$
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NB: (iii) \( \implies \) the 2-surfaces \( S_t \) are Lie-dragged by \( h \)

Define \( C := \frac{1}{2} h \cdot h \)

\( \mathcal{H} \) is spacelike \( \iff \ C > 0 \iff h \) is spacelike
\( \mathcal{H} \) is null \( \iff \ C = 0 \iff h \) is null
\( \mathcal{H} \) is timelike \( \iff \ C < 0 \iff h \) is timelike.
Normal null frame associated with the evolution vector

The foliation \((S_t)_{t \in \mathbb{R}}\) entirely fixes the ambiguities in the choice of the null normal frame \((\ell, k)\), via the evolution vector \(h\):
there exists a unique normal null frame \((\ell, k)\) such that

\[ h = \ell - Ck \quad \text{and} \quad \ell \cdot k = -1 \]

Normal fundamental form:

\[ \Omega^{(\ell)} := -k \cdot \nabla_q \ell \quad \text{or} \quad \Omega^{(\ell)}_{\alpha} := -k_\mu \nabla_\nu \ell^\mu q^\nu_{\alpha} \]

Evolution of \(h\) along itself:

\[ \nabla_h h = \kappa \ell + (C\kappa - \mathcal{L}_h C)k - \mathcal{D}C \]

NB: null limit : \(C = 0, \ h = \ell \implies \nabla_\ell \ell = \kappa \ell \implies \kappa = \text{surface gravity} \]
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4. Applications to numerical relativity
Hartle and Hawking (1972, 1973): introduced the concept of black hole viscosity when studying the response of the event horizon to external perturbations.

Damour (1979): 2-dimensional Navier-Stokes like equation for the event horizon $\implies$ shear viscosity and bulk viscosity.

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Damour (1979): 2-dimensional **Navier-Stokes** like equation for the event horizon $\Rightarrow$ *shear viscosity* and *bulk viscosity*

Thorne and Price (1986): **membrane paradigm** for black holes

Shall we restrict the analysis to the event horizon?

Can we extend the concept of viscosity to the local characterizations of black hole recently introduced, i.e. *future outer trapping horizons* and *dynamical horizons*?

**NB:** *event horizon* = null hypersurface  
*future outer trapping horizon* = null or spacelike hypersurface  
*dynamical horizon* = spacelike hypersurface
**Viscous fluid analogy**

**Original Damour-Navier-Stokes equation**

*Hyp:* $\mathcal{H} =$ null hypersurface (particular case: black hole **event horizon**)  

Then $h = \ell$ ($C = 0$)

Damour (1979) has derived from **Einstein equation** the relation

\[
S\mathcal{L}_\ell \Omega^{(\ell)} + \theta^{(\ell)} \Omega^{(\ell)} = \mathcal{D}_\kappa - \mathcal{D} \cdot \sigma^{(\ell)} + \frac{1}{2} \mathcal{D} \theta^{(\ell)} + 8\pi \vec{q}^* T \cdot \ell
\]

or equivalently

\[
S\mathcal{L}_\ell \pi + \theta^{(\ell)} \pi = -\mathcal{D} P + 2\mu \mathcal{D} \cdot \sigma^{(\ell)} + \zeta \mathcal{D} \theta^{(\ell)} + f \quad (*)
\]

with

- $\pi := -\frac{1}{8\pi} \Omega^{(\ell)}$ momentum surface density
- $P := \frac{\kappa}{8\pi}$ pressure
- $\mu := \frac{1}{16\pi}$ shear viscosity
- $\zeta := -\frac{1}{16\pi}$ bulk viscosity
- $f := -\vec{q}^* T \cdot \ell$ external force surface density ($T =$ stress-energy tensor)
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$$S\mathcal{L}_\ell \Omega^{(\ell)} + \theta^{(\ell)} \Omega^{(\ell)} = D\kappa - D \cdot \sigma^{(\ell)} + \frac{1}{2} D\theta^{(\ell)} + 8\pi \vec{q}^* T \cdot \ell$$

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\textbf{(*) is identical to a 2-dimensional Navier-Stokes equation}
This is only an analogy with hydrodynamics
to avoid any misunderstanding...

This is only an analogy with hydrodynamics because

“A black hole is not a water fall” (Clifford Will)
Viscous fluid analogy

Negative bulk viscosity of event horizons

From the Damour-Navier-Stokes equation, \( \zeta = -\frac{1}{16\pi} < 0 \)

This negative value would yield to a *dilation or contraction instability* in an ordinary fluid.

It is in agreement with the tendency of a null hypersurface to continually contract or expand.

The event horizon is stabilized by the *teleological condition* imposing its expansion to vanish in the far future (equilibrium state reached).
Starting remark: in the null case (event horizon), $\ell$ plays two different roles:

- evolution vector along $\mathcal{H}$ (e.g. term $\mathcal{S}\mathcal{L}_\ell$)
- normal to $\mathcal{H}$ (e.g. term $\vec{q}^* T \cdot \ell$)

When $\mathcal{H}$ is no longer null, these two roles have to be taken by two different vectors:

- evolution vector: obviously $h$
- vector normal to $\mathcal{H}$: a natural choice is $m := \ell + Ck$
Viscous fluid analogy

Generalized Damour-Navier-Stokes equation

From the contracted Ricci identity applied to the vector $m$ and projected onto $S_t$:

$$(\nabla_\mu \nabla_\nu m^\mu - \nabla_\nu \nabla_\mu m^\mu) q^\nu_\alpha = R_{\mu\nu} m^\mu q^\nu_\alpha$$

and using Einstein equation to express $R_{\mu\nu}$, one gets an evolution equation for $\Omega^{(\ell)}$ along $\mathcal{H}$:

$$S\mathcal{L}_h \Omega^{(\ell)} + \theta^{(h)} \Omega^{(\ell)} = \mathcal{D}\kappa - \mathcal{D} \cdot \sigma^{(m)} + \frac{1}{2} \mathcal{D}\theta^{(m)} - \theta^{(k)} \mathcal{D}C + 8\pi \vec{q}^* T \cdot m$$

- $\Omega^{(\ell)}$: normal fundamental form of $S_t$ associated with null normal $\ell$
- $\theta^{(h)}$, $\theta^{(m)}$ and $\theta^{(k)}$: expansion scalars of $S_t$ along the vectors $h$, $m$ and $k$ respectively
- $\mathcal{D}$: covariant derivative within $(S_t, q)$
- $\kappa$: component of $\nabla_h h$ along $\ell$
- $\sigma^{(m)}$: shear tensor of $S_t$ along the vector $m$
- $C$: half the scalar square of $h$
If $\mathcal{H}$ is a null hypersurface,

$$h = m = \ell \quad \text{and} \quad C = 0$$

and we recover the original Damour-Navier-Stokes equation:

$$S\mathcal{L}_{\ell} \Omega^{(\ell)} + \theta^{(\ell)} \Omega^{(\ell)} = \mathcal{D}_{\mathcal{H}} - \mathcal{D} \cdot \sigma^{(\ell)} + \frac{1}{2} \mathcal{D} \theta^{(\ell)} + 8\pi \vec{q}^* T \cdot \ell$$
Viscous fluid analogy

Case of future trapping horizons

Definition [Hayward, PRD 49, 6467 (1994)]:

$\mathcal{H}$ is a **future trapping horizon** iff $\theta(\ell) = 0$ and $\theta(k) < 0$.

The generalized Damour-Navier-Stokes equation reduces then to

$$S_{L_h} \Omega^{(\ell)} + \theta^{(h)} \Omega^{(\ell)} = D\kappa - D \cdot \sigma^{(m)} - \frac{1}{2} D\theta^{(h)} - \theta^{(k)} DC + 8\pi \vec{q}^* T \cdot m$$

[EG, PRD 72, 104007 (2005)]

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The explanation: it is $\theta^{(m)}$ which appears in the general equation and

$$
\theta^{(m)} + \theta^{(h)} = 2\theta^{(\ell)} \implies \begin{cases} 
\text{event horizon} (m = h) : & \theta^{(m)} = \theta^{(\ell)} \\
\text{trapping horizon} (\theta^{(\ell)} = 0) : & \theta^{(m)} = -\theta^{(h)}
\end{cases}
$$
Viscous fluid form

\[ S\mathcal{L}_h \pi + \theta^{(h)} \pi = -D P + \frac{1}{8\pi} D \cdot \sigma^{(m)} + \zeta D \theta^{(h)} + f \]

with \( \pi := -\frac{1}{8\pi} \Omega^{(\ell)} \) momentum surface density

\( P := \frac{\kappa}{8\pi} \) pressure

\( \frac{1}{8\pi} \sigma^{(m)} \) shear stress tensor

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\( f := -\vec{q}^* T \cdot m + \frac{\theta^{(k)}}{8\pi} \mathcal{D} C \) external force surface density

Similar to the Damour-Navier-Stokes equation for an event horizon, except

- the **Newtonian-fluid** relation between *stress* and *strain* does not hold:

  \[ \sigma^{(m)}/8\pi \neq 2\mu \sigma^{(h)}, \text{ rather } \sigma^{(m)}/8\pi = [\sigma^{(h)} + 2C \sigma^{(k)}]/8\pi \]
Viscous fluid analogy

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**a post-Newtonian fluid?**
Viscous fluid form

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  a post-Newtonian fluid?

- positive bulk viscosity

This positive value of bulk viscosity shows that FOTHs and DHs behave as “ordinary” physical objects, in perfect agreement with their local nature.
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Angular momentum of trapping horizons

**Definition** [Booth & Fairhurst, CQG 22, 4545 (2005)]: Let \( \varphi \) be a vector field on \( \mathcal{H} \) which

- is tangent to \( S_t \)
- has closed orbits
- has vanishing divergence with respect to the induced metric: \( \mathcal{D} \cdot \varphi = 0 \)
  (weaker than being a Killing vector of \( (S_t, q) \) !)

For dynamical horizons, \( \theta^{(h)} \neq 0 \) and there is a unique choice of \( \varphi \) as the generator (conveniently normalized) of the curves of constant \( \theta^{(h)} \)

[Hayward, PRD 74, 104013 (2006)]

The *generalized angular momentum associated with* \( \varphi \) is then defined by

\[
J(\varphi) := -\frac{1}{8\pi} \oint_{S_t} \langle \Omega^{(\ell)}, \varphi \rangle s \epsilon,
\]

**Remark 1:** does not depend upon the choice of null vector \( \ell \), thanks to the divergence-free property of \( \varphi \)

**Remark 2:**

- coincides with Ashtekar & Krishnan’s definition for a dynamical horizon
- coincides with Brown-York angular momentum if \( \mathcal{H} \) is timelike and \( \varphi \) a Killing vector
Angular momentum and area evolution laws

Angular momentum flux law

Under the supplementary hypothesis that $\phi$ is transported along the evolution vector $h$, $\mathcal{L}_h \phi = 0$, the generalized Damour-Navier-Stokes equation leads to

$$\frac{d}{dt} J(\phi) = - \oint_{S_t} T(m, \phi) s^\epsilon - \frac{1}{16\pi} \oint_{S_t} \left[ \sigma(m) : \mathcal{L}_\phi q - 2\theta^k \phi \cdot D\mathcal{C} \right] s^\epsilon$$

[EG, PRD 72, 104007 (2005)]
Angular momentum and area evolution laws

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Two interesting limiting cases:
Angular momentum and area evolution laws

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Two interesting limiting cases:

- **\( H = \) null hypersurface :** \( C = 0 \) and \( m = \ell : \)
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  \]
  i.e. Eq. (6.134) of the *Membrane Paradigm* book (Thorne, Price & MacDonald 1986)

Eric Gourgoulhon (LUTH)
Modelling black holes as trapping horizons
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- **\( \mathcal{H} = \) future trapping horizon :**
  
  \[
  \frac{d}{dt} J(\varphi) = - \int_{S_t} T(m, \varphi)^S \epsilon - \frac{1}{16\pi} \int_{S_t} \sigma^{(m)} : \mathcal{L}_\varphi q^S \epsilon
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Angular momentum and area evolution laws

Area evolution law for an event horizon

\( A(t) \) : area of the 2-surface \( S_t \); \( ^s\epsilon \) : volume element of \( S_t \); \( \bar{\kappa}(t) := \frac{1}{A(t)} \int_{S_t} \kappa \, ^s\epsilon \)

Integrating the null Raychaudhuri equation on \( S_t \), one gets

\[
\frac{d^2 A}{dt^2} - \bar{\kappa} \frac{dA}{dt} = - \int_{S_t} \left[ 8\pi T(\ell, \ell) + \sigma^{(\ell)} : \sigma^{(\ell)} - \frac{(\theta^{(\ell)})^2}{2} + (\bar{\kappa} - \kappa)\theta^{(\ell)} \right] \, ^s\epsilon
\]

(1) [Damour, 1979]
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(1)

[Damour, 1979]

Simplified analysis : assume \( \bar{\kappa} = \text{const} > 0 \) :

- Cauchy problem \( \Rightarrow \) diverging solution of the homogeneous equation:
  \[
  \frac{dA}{dt} = \alpha \exp(\bar{\kappa}t)
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Angular momentum and area evolution laws

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  \[ \frac{dA}{dt} = \alpha \exp(\bar{\kappa}t) \]

- Correct treatment: impose \( \frac{dA}{dt} = 0 \) at \( t = +\infty \) (teleological !)
  \[ \frac{dA}{dt} = \int_{t}^{+\infty} D(u)e^{\bar{\kappa}(t-u)} \, du \quad D(t) : \text{r.h.s. of Eq. (1)} \]

Non causal evolution
Area evolution law for a dynamical horizon

Dynamical horizon: \( C > 0; \quad \kappa' := \kappa - \mathcal{L}_h \ln C; \quad \bar{\kappa}'(t) := \frac{1}{A(t)} \int_{S_t} \kappa' s \epsilon \)

From the \((m, h)\) component of Einstein equation, one gets

\[
\frac{d^2 A}{dt^2} + \bar{\kappa}' \frac{dA}{dt} = \int_{S_t} \left[ 8\pi T(m, h) + \sigma^{(h)} : \sigma^{(m)} + \frac{(\theta^{(h)})^2}{2} + (\bar{\kappa}' - \kappa')\theta^{(h)} \right] s \epsilon
\]

\[ (2) \]

[EG & Jaramillo, PRD 74, 087502 (2006)]
Angular momentum and area evolution laws

Area evolution law for a dynamical horizon

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[EG & Jaramillo, PRD 74, 087502 (2006)]

Simplified analysis: assume $\bar{\kappa}' = \text{const} > 0$

(OK for small departure from equilibrium [Booth & Fairhurst, PRL 92, 011102 (2004)])

Standard Cauchy problem:

$$\frac{dA}{dt} = \frac{dA}{dt} \bigg|_{t=0} + \int_0^t D(u)e^{\bar{\kappa}'(u-t)} du \quad D(t) : \text{r.h.s. of Eq. (2)}$$

Causal evolution, in agreement with local nature of dynamical horizons
Outline

1. Local approaches to black holes
2. Viscous fluid analogy
3. Angular momentum and area evolution laws
4. Applications to numerical relativity
Applications to numerical relativity

- **Initial data**: isolated horizons (helical symmetry)
  - [EG, Grandclément & Bonazzola, PRD 65, 044020 (2002)]
  - [Grandclément, EG & Bonazzola, PRD 65, 044021 (2002)]
  - [Cook & Pfeiffer, PRD 70, 104016 (2004)]

- **A posteriori analysis**: estimating mass, linear and angular momentum of formed black holes
  - [Schnetter, Krishnan & Beyer, PRD 74, 024028 (2006)]
  - [Cook & Whiting, PRD 76, 041501 (2007)]
  - [Krishnan, Lousto & Zlochower, PRD 76, 081501(R) (2007)]

- **Numerical construction of spacetime**: inner boundary conditions for a constrained scheme with “black hole excision”
  - [Jaramillo, EG, Cordero-Carrión, & J.M. Ibáñez, PRD 77, 047501 (2008)]
A few words about the history of 3+1 formalism

- **G. Darmois (1927)**: 3+1 Einstein equations in terms of \((\gamma_{ij}, K_{ij})\) with \(\alpha = 1\) and \(\beta^i = 0\) (Gaussian normal coordinates)
  Cauchy problem well posed for *analytic* initial data
- **A. Lichnerowicz (1939)**: \(\alpha \neq 1\) and \(\beta^i = 0\) (normal coordinates)
- **Y. Choquet-Bruhat (1948)**: \(\alpha \neq 1\) and \(\beta^i \neq 0\) (general coordinates)
- **Y. Choquet-Bruhat (1952)**: Cauchy problem well posed for *smooth* (i.e. generic) initial data
- **R. Arnowitt, S. Deser & C.W. Misner (1962)**: *Hamiltonian formulation* of GR based on a 3+1 decomposition in terms of \((\gamma_{ij}, \pi^{ij})\)
  *NB*: spatial projection of *Einstein tensor* instead of *Ricci tensor* in previous works
- **J. Wheeler (1964)**: coined the terms *lapse* and *shift*
- **J.W. York (1979)**: modern 3+1 decomposition based on spatial projection of *Ricci tensor*