Gravitational waves in LISA band from bodies orbiting the Galactic Center black hole

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based on arXiv:1903.02049

Workshop on wave forms
GdR Ondes gravitationnelles
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The black hole Sgr A* at the Galactic center

- distance: \( d = 8.12 \) kpc
- mass:
  \[
  M = 4.10 \times 10^6 M_\odot \\
  = 20.2 \text{ s} \\
  = 6.06 \times 10^9 \text{ m} \\
  = 4.05 \times 10^{-2} \text{ au} \\
  = 1.96 \times 10^{-7} \text{ pc}
  \]
  \( \iff \)
  \( 1 \text{ pc} = 5.10 \times 10^6 M \)
- spin \( J = aM \) unknown yet...

← Orbit of star S2 around Sgr A*

[GRAVITY team, A&A 615, L15 (2018)]
GW frequencies from circular orbits around Sgr A*

Angular velocity of circular equatorial orbits around a Kerr BH

$$\omega_0 = \frac{M^{1/2}}{r_0^{3/2} + aM^{1/2}}$$

Dominant GW frequency

$$f_{m=2} = \frac{\omega_0}{\pi}$$

Sgr A* mass

$$M = 4.10 \times 10^6 \, M_\odot = 20.2 \, \text{s}$$

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\[ M = 4.10 \times 10^6 \, M_\odot \]
\[ = 20.2 \, \text{s} \]

Roche radius:

\[ r_R \simeq 1.14 \left( \frac{M}{\rho} \right)^{1/3} \]

Frequencies of Sgr A* close orbits are in LISA band

ISCO for $a = M$: $f_{m=2} = 7.9$ mHz
Frequencies of Sgr A* close orbits are in LISA band

ISCO for $a = M$: \( f_{m=2} = 7.9 \text{ mHz} \) \( \leftarrow \) coincides with LISA max. sensitivity!
Previous studies of Sgr A* as a source for LISA

- Freitag (2003) [ApJ 583, L21]: GW from orbiting stars at quadrupole order; low-mass main-sequence (MS) stars are good candidates for LISA

- Barack & Cutler (2004) [PRD 69, 082005]: $0.06M_\odot$ MS star observed $10^6$ yr before plunge $\Rightarrow$ SNR = 11 in 2 yr of LISA data $\Rightarrow$ Sgr A*'s spin within 0.5% accuracy

- Berry & Gair (2013) [MNRAS 429, 589]: extreme-mass-ratio burst (single periastron passage on a highly eccentric orbit) $\Rightarrow$ GW burst $\Rightarrow$ LISA detection of $10M_\odot$ for periastron $< 65M_\odot$; event rate could be $\sim 1$ yr$^{-1}$

- Linial & Sari (2017) [MNRAS 469, 2441]: GW from orbiting MS stars undergoing Roche lobe overflow $\Rightarrow$ detectability by LISA; possibility of a reverse chirp signal (outspiral)

- Kühnel et al. (2018) [arXiv:1811.06387]: GW from an ensemble of macroscopic dark matter candidates orbiting Sgr A*, such as primordial BHs, with masses in the range $10^{-13} - 10^3 M_\odot$

- Amaro-Seoane (2019) [arXiv:1903.10871]: Extremely Large Mass-Ratio Inspirals (X-MRI) $\Rightarrow$ brown dwarfs orbiting Sgr A* should be detected in great numbers by LISA: $\sim 20$ in band at any time
All previous studies have been performed in a Newtonian framework (quadrupole formula), except that of Barack & Cutler (2004), which is post-Newtonian. Now, for orbits close to the ISCO, relativistic effects are expected to be important.

⇒ we have adopted a **fully relativistic framework**:

- Sgr A* is modeled as a Kerr BH and GW are computed via the theory of perturbations of the Kerr metric
- tidal effects are evaluated via the theory of Roche potential in the Kerr metric developed by Dai & Blandford (2013) [MNRAS 434, 2948]
Our study

All previous studies have been performed in a Newtonian framework (quadrupole formula), except that of Barack & Cutler (2004), which is post-Newtonian. Now, for orbits close to the ISCO, relativistic effects are expected to be important. 

⇒ we have adopted a fully relativistic framework:

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Limitation: circular equatorial orbits; valid for

- inspiralling compact objects from the tidal disruption of a binary (zero-eccentricity EMRI)
- MS stars formed in an accretion disk
- compact objects resulting from the most massive of such stars
- ∼ 1/4 of the population of brown dwarfs studied by Amaro-Seoane (2019)
Waveforms from circular orbits computed as linear perturbations of Kerr metric (Teukolsky 1973)

Detweiler (1978)

\[ h_+ - ih_\times = \frac{2\mu}{r} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Z_{\ell m}^\infty(r_0)}{(m\omega_0)^2} -2S_{\ell m}^{a m \omega_0}(\theta, \varphi) e^{-im(\omega_0(t-r_*)+\varphi_0)} \]

- \( \mu \): mass of orbiting object; \((t, r, \theta, \varphi)\): Boyer-Lindquist coordinates of the observer
- \(-2S_{\ell m}^{a m \omega_0}(\theta, \varphi)\): spheroidal harmonics of spin weight \(-2\)
Waveforms from circular orbits  
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\[-2S_{\ell m}^{am\omega_0}(\theta, \varphi): \text{spheroidal harmonics of spin weight } -2\]

Example for \(a = 0.9\ M\), \(r_0 = r_{\text{ISCO}}(a)\) and viewing angle \(\theta = 0\) (face-on)
Waveforms from circular orbits computed as linear perturbations of Kerr metric (Teukolsky 1973)

Detweiler (1978)

\[
\begin{align*}
    h_+ - i h_\times &= \frac{2\mu}{r} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Z_{\ell m}(r_0)}{(m\omega_0)^2} \mathcal{S}_{\ell m}^{am\omega_0}(\theta, \varphi) e^{-i m(\omega_0(t-r_*) + \varphi_0)}\end{align*}
\]

\(\mu\): mass of orbiting object; 
\((t, r, \theta, \varphi)\): Boyer-Lindquist coordinates of the observer

\(-2\mathcal{S}_{\ell m}^{am\omega_0}(\theta, \varphi)\): spheroidal harmonics of spin weight \(-2\)

Example for \(a = 0.9 \, M\), \(r_0 = r_{\text{ISCO}}(a)\) and viewing angle \(\theta = \pi/4\)

\[\begin{align*}
    &a = 0.90 \, M, \quad r_0 = 2.321 \, M, \quad \theta = \frac{\pi}{4} \\
    &\frac{r h_m}{\mu} \\
    &\frac{r h}{\mu}
\end{align*}\]
Waveforms from circular orbits
computed as linear perturbations of Kerr metric (Teukolsky 1973)

Detweiler (1978)

\[
\begin{align*}
  h_+ - ih_\times &= \frac{2\mu}{r} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Z_{\ell m}(r_0)}{(m\omega_0)^2} -2S_{\ell m}^{am\omega_0} (\theta, \varphi) e^{-im(\omega_0(t-r_*)+\varphi_0)}
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Example for \(a = 0.9 \, M\), \(r_0 = r_{\text{ISCO}}(a)\) and viewing angle \(\theta = \pi/2\) (edge-on)
Implementation: the kerrgeodesic_gw package

All computations (GW waveforms, SNR in LISA, energy fluxes, inspiralling time, etc.) have been implemented as a Python package for the open-source mathematics software system SageMath:

**kerrgeodesic_gw**

- entirely open-source:
  - https://github.com/BlackHolePerturbationToolkit/kerrgeodesic_gw
- is distributed via the PyPi (the Python Package Index):
  - https://pypi.org/project/kerrgeodesic-gw/
  so that the installation in SageMath is very easy:
  - sage -pip install kerrgeodesic_gw
- is part of the *Black Hole Perturbation Toolkit*:
  - http://bhptoolkit.org/
Signal-to-noise ratio in the LISA detector

\[ \text{SNR: } \rho \times \left( \frac{1 M^\odot}{\mu} \right) \left( \frac{1 \text{d}}{T} \right)^{1/2} \]

- \( \theta = 0 \)
- \( \theta = \pi/4 \)
- \( \theta = \pi/2 \)

- \( a = 0 \)
- \( a = 0.50 M \)
- \( a = 0.90 M \)
- \( a = 0.98 M \)

Signal-to-noise ratio in the LISA detector

\[ \text{SNR: } \rho \times \left( \frac{1 \, M_\odot}{\mu} \right) \left( \frac{1 \text{d}}{T} \right)^{1/2} \]

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- \( a = 0.50 \, M \)
- \( a = 0.90 \, M \)
- \( a = 0.98 \, M \)

Signal-to-noise ratio in the LISA detector

\[ \text{SNR: } \rho \times \left( \frac{1M_\odot}{\mu} \right) \left( \frac{1d}{T} \right)^{1/2} \]

\[ r_0/M \]

\[ \theta = 0 \] \quad \theta = \pi/4 \quad \theta = \pi/2 \]

- \( a = 0 \)
- \( a = 0.50 \, M \)
- \( a = 0.90 \, M \)
- \( a = 0.98 \, M \)
- Roche limit for \( 1M_\odot \)-star/Jupiter
- Roche limit for Earth-type
- Roche limit for \( 0.3M_\odot \)-star
- Roche limit for brown dwarf

\[ \text{[Gourgoulhon, Le Tiec, Vincent & Warburton, arXiv:1903.02049]} \]
Minimal detectable mass by LISA

Detection criteria: SNR $\geq 10$
Observation time: 1 yr

SNR=10 ($T = 1 \text{ yr}$)

- $\theta = 0$
- $\theta = \pi/4$
- $\theta = \pi/2$

$\mu_{\text{min}} [M_{\odot}]$

$r_0/M$

- $a = 0$
- $a = 0.50 M$
- $a = 0.90 M$
- $a = 0.98 M$
- Roche limit for $1 M_{\odot}$-star/Jupiter
- Roche limit for rocky body
- Roche limit for $0.2 M_{\odot}$-star
- Roche limit for brown dwarf
Maximum orbital radius $r_{0,\text{max}}$ for a SNR = 10 detection by LISA in one year of data, as a function of the mass $\mu$ of the object orbiting around Sgr A*.
$T_{\text{life}}$: time for a compact object to reach the ISCO on the slow inspiral induced by gravitational radiation reaction
Time spent in LISA band

Inspiral time from orbit \( r_0 \) to orbit \( r_1 \) due to reaction to gravitational radiation:

\[
T_{\text{ins}}(r_0, r_1) = \frac{M^2}{2\mu} \int_{r_1/M}^{r_0/M} \frac{1 - 6/x + 8\tilde{a}/x^{3/2} - 3\tilde{a}^2/x^2}{(1 - 3/x + 2\tilde{a}/x^{3/2})^{3/2}} \frac{dx}{x^2(\tilde{L}_\infty(x) + \tilde{L}_H(x))}
\]

where \( \tilde{L}_\infty, H(x) := (M/\mu)^2 L_\infty, H(xM) \) and \( L_\infty \) (resp. \( L_H \)) is the total GW power emitted at infinity (resp. through the BH event horizon) by a particle of mass \( \mu \) orbiting at \( r = xM \)

**Compact object**

\[
T_{\text{in-band}} = T_{\text{ins}}(r_{0,\text{max}}, r_{\text{ISCO}}) = T_{\text{life}}(r_{0,\text{max}})
\]

**MS stars and brown dwarfs**

\[
T_{\text{in-band}} \geq T_{\text{in-band}}^{\text{ins}} = T_{\text{ins}}(r_{0,\text{max}}, r_{\text{Roche}})
\]
Time in LISA band for an inspiralling compact object

\[ T_{\text{in-band}} \text{[yr]} \]

\[ \mu \text{[} M_\odot \text{]} \]

\[ \theta = 0 \quad \theta = \pi/4 \quad \theta = \pi/2 \]

\[ a = 0 \quad a = 0.98 M \]

Gourgoulhon, Le Tiec, Vincent & Warburton (shortinst)
Results for
- inclination angle $\theta = 0$
- BH spin $a = 0$ (outside parentheses) and $a = 0.98M$ (inside parentheses)

<table>
<thead>
<tr>
<th></th>
<th>brown dwarf</th>
<th>red dwarf</th>
<th>Sun-type</th>
<th>2.4 $M_\odot$-star</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu/M_\odot$</td>
<td>0.062</td>
<td>0.20</td>
<td>1</td>
<td>2.40</td>
</tr>
<tr>
<td>$\rho/\rho_\odot$</td>
<td>131.</td>
<td>18.8</td>
<td>1</td>
<td>0.367</td>
</tr>
<tr>
<td>$r_{0,\text{max}}/M$</td>
<td>28.2 (28.0)</td>
<td>35.0 (34.9)</td>
<td>47.1 (47.0)</td>
<td>55.6 (55.6)</td>
</tr>
<tr>
<td>$f_{m=2}(r_{0,\text{max}})$ [mHz]</td>
<td>0.105 (0.106)</td>
<td>0.076 (0.076)</td>
<td>0.049 (0.049)</td>
<td>0.038 (0.038)</td>
</tr>
<tr>
<td>$r_{\text{Roche}}/M$</td>
<td>7.31 (6.93)</td>
<td>13.3 (13.0)</td>
<td>34.2 (34.1)</td>
<td>47.6 (47.5)</td>
</tr>
<tr>
<td>$T_{\text{in-band}}^{\text{ins}}$ [$10^5$ yr]</td>
<td>4.98 (5.55)</td>
<td>3.72 (3.99)</td>
<td>1.83 (1.89)</td>
<td>0.938 (0.945)</td>
</tr>
</tbody>
</table>
What about the accretion flow?

 Orbital motion of a flare at $r_0 \sim 7M$

 observed by GRAVITY

[GRAVITY team, A&A 618, L10 (2018)]
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Total mass of the accretion flow:

$\sim 10^{-11} M_\odot$

$R=7 R_g \ a=0 \ i=160^\circ \ \Omega=160^\circ \ \chi_r^2=1.2$
What about the accretion flow?

Orbital motion of a flare at $r_0 \sim 7M$

observed by GRAVITY

\[ R = 7 \, R_g \quad a = 0 \quad i = 160^\circ \quad \Omega = 160^\circ \quad \chi_r^2 = 1.2 \]

Total mass of the accretion flow:

$\sim 10^{-11} M_\odot$

\[ \rightarrow \text{inhomogeneities (such as flares) not detectable by LISA} \]

[GRAVITY team, A&A 618, L10 (2018)]
Conclusions

- We have computed GW emission and SNR in LISA for close circular orbits around Sgr A* in full general relativity.
- The time spent in LISA band (SNR ≥ 10) during the slow inspiral has been evaluated.
- All computations have been implemented in an open-source SageMath package, `kerrgeodesic_gw`, as part of the Black Hole Perturbation Toolkit.
- LISA has the capability to detect orbiting masses close to the ISCO as small as \( \sim 10M_{\text{Earth}} \) or even \( \sim 1M_{\text{Earth}} \) if Sgr A* is a fast rotator \((a \geq 0.9M)\); this could involve primordial BHs or very dense artificial objects.
- White dwarfs, NSs, stellar BHs, BHs of mass \( \geq 10^{-4}M_\odot \), MS stars of mass \( \leq 2.5M_\odot \) and brown dwarfs orbiting Sgr A* are all detectable in 1 yr of LISA data with SNR \( \geq 10 \).
- The longest times in-band, of the order of \( 10^6 \) years, are achieved for primordial BHs of mass \( \sim 10^{-3}M_\odot \) down to \( 10^{-5}M_\odot \), depending on the spin of Sgr A*, as well as for brown dwarfs, just followed by white dwarfs and low mass MS stars.