The best chirplet chain for the detection of gravitational wave chirps

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   Exploratory searches, why?
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3 Conciliate both viewpoints
   Chirplet chains, Phys. Rev. D73, 042003, 2006
Targeted search and matched filtering

notation: sampled signals, \( \mathbf{x} \equiv \{x_k \equiv x(k/f_s), k = 0 \ldots N - 1 \} \)

detection = decide which hypothesis fits the data
Targeted search and matched filtering

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$(H_0)$ $x_k = n_k$  white Gaussian noise

$(H_1)$ $x_k = s_k + n_k$  signal+noise

define a statistic $\lambda(\mathbf{x}) \leq \eta \leftrightarrow$ choose $H_0$ or $H_1$, partition (here, of $\mathbb{R}^N$)
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\( (H_1) \quad x_k = s_k + n_k \quad \text{signal+noise} \)

define a statistic \( \lambda(x) \leq \eta \leftrightarrow \) choose \( H_0 \) or \( H_1 \), partition (here, of \( \mathbb{R}^N \))

which \( \lambda \) is best? criterion: Neymann-Pearson (NP), error prob.

minimize \( \mathbb{P}(\lambda(x) < \eta|H_1) \) for a given \( \mathbb{P}(\lambda(x) > \eta|H_0) \)

solution = likelihood ratio \( \lambda(x) = \frac{\mathbb{P}(\mathbf{x}|H_1)}{\mathbb{P}(\mathbf{x}|H_0)} \)
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solution = likelihood ratio $\lambda(\mathbf{x}) = \frac{P(\mathbf{x} | H_1)}{P(\mathbf{x} | H_0)}$

for our problem, $\log(\lambda(\mathbf{x})) \propto \|\mathbf{x} - \mathbf{s}\|^2_2 - \|\mathbf{x}\|^2_2$

simplify: $\ell(\mathbf{x}) = \langle \mathbf{x}, \mathbf{s} \rangle = \sum_{k=0}^{N-1} x_k s_k$

matched filter: correlation of the data with a template $\mathbf{s}$
**Unknown parameters and bank of matched filters**

when the signal $s$ depends on unknown parameters $p$ . . .

likelihood ratio: $\lambda(x; p) = \frac{\mathbb{P}(x|H_1, p)}{\mathbb{P}(x|H_0)}$

NP uniformly for all values of $p$? no solution in general
**Unknown parameters and bank of matched filters**

when the signal \( s \) depends on unknown parameters \( p \) ...

\[
\text{likelihood ratio: } \lambda(x; p) = \frac{\mathbb{P}(x|H_1, p)}{\mathbb{P}(x|H_0)}
\]

NP uniformly for all values of \( p \)? no solution in general

sub-optimal but works well: “generalized likelihood ratio test”

idea: replace \( p \) by an (ML) estimate \( \hat{p} = \arg\max_p \lambda(x; p) \)

two ways for doing this

1. if analytical expression \( \hat{p}(x) \) exists, replace \( \ell(x) = \lambda(x; \hat{p}(x)) \)

2. if not, maximize numerically (exhaustive search):
   \( \ell(x) = \max_p \lambda(x; p) \)

for our problem, this is a bank of matched filters
targeted search = matched filter bank obtained from Physics

Chassande-Mottin  BCC
Why exploratory searches?

- targeted search is sensitive, strength and also weakness
  - require reliable and precise model
  - does not incorporate model uncertainties
- data are precious (expensive!): get the most from them
  - look for speculative or unknown sources!
- moral: be exploratory! let the model be more “general” . . .
  - relax assumptions = increase robustness
- . . . but not too general! be “quasi-physical”
  - exclude non-feasible/unlikely candidates
  - use “good sense” assumption to restrict the model

note: exploration useful for detection, not for identification and interpretation which needs complete physical model!
GW unmodeled chirps: motivations

- **basic idea:** GW = system “radiates away its asymmetries” if *orbiting and slowly moving* → quasi-periodic GWs (=chirps)

GW chirps are generic signatures of orbiting systems

- **this information is robust:** this remains true even we don’t know the system dynamics in detailed.

- **consequence:** search for chirps in “general”
GW (unmodeled) chirps: generic model

- generic model for chirps
  
  **GW chirps**: \( s(t) \equiv A \cos(\phi(t) + \varphi_0) \)

  unknown amplitude \( A \) and initial phase \( \varphi_0 \)
GW (unmodeled) chirps: generic model

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  \[ s(t) \equiv A \cos(\phi(t) + \varphi_0) \]
  unknown amplitude \( A \) and initial phase \( \varphi_0 \)

- unknown phase evolution \( \phi(t) \)
  exclude non-physical with “good sense” constraint:
  impose \( |\dot{f}(t)| \leq F' \) and \( |\ddot{f}(t)| \leq F'' \) where \( f(t) = (2\pi)^{-1}\dot{\phi}(t) \).
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- typical duration \( T \sim \) few sec in detector band
Chirps in the time-frequency plane (1)

**heuristic:** chirp = “filiform” pattern in time-frequency plane
Two degrees of freedom (2)

which TF representation?
spectrogram, wavelets, Wigner-Ville, Cohen, reassignment, etc.

which pattern search?
Hough, “crazy climbers”, “snakes”, road tracker in satellite images, etc.
Multiple approaches... (3)

- Morvidone & Torrésani, *IJWMIP*, 2003
- Carmona, Hwang & Torrésani, *IEEE SP*, 1998
- Pinto et al., *Proc. of GWDAW*, 1997
- Innocent & Torrésani, *ACHA*, 1997
Chirps and quadrature matched filtering

Let us apply generalized likelihood ratio test to chirps.

We have 3 unknown parameters $\mathbf{p} = \{A, \varphi_0, \phi(t)\}$.

Two simple ones $A, \varphi_0 = \text{analytical replacement}$

$$\log(\max_{A,\varphi_0} \lambda) \propto \left| \sum_{k=0}^{N-1} x_k \exp i \phi_k \right|^2 \equiv \ell(x, \phi) \leq \eta$$

Quadrature matched filtering
When chirp phase is not known... (2)

- last unknown parameter $\phi(t)$, how to do $\max_\phi \ell(x, \phi)$?
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we “receive” $x_k \triangleq A \cos(\phi_k + \varphi_0)$ and we “search” with template $\phi_k^*$

$$\Delta \ell(\phi, \phi^*) \equiv \frac{\ell(s, \phi) - \ell(s, \phi^*)}{\ell(s, \phi)}$$

the distance between two grid nodes should be small
**Conciliate viewpoints?**

Does this method apply in general?

1. can we build a bank of matched filters for GW chirps?
2. with which templates?
Chirplet chains (CC), Phys. Rev. D73, 042003

CCs are piecewise linear chirps

free parameters: $N_t$, $N_f$, $N'_r$, $N''_r$
CCs form a tight template grid

if $N'_r$ and $N''_r$ are large enough, for all smooth chirp $\phi$, there exists a CC $\phi^*$ such that

$$\Delta \ell(\phi, \phi^*) \lesssim C \left[ \frac{1}{2} \left( \frac{\sqrt{3} F'' T^3}{N_t} \right)^2 + \frac{1}{2} \left( \frac{2N}{N_f} \right) \right]^2$$
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CC grid is tight!
CCs form a tight template grid

if \( N_r' \) and \( N_r'' \) are large enough, for all smooth chirp \( \phi \), there exists a CC \( \phi^* \) such that

\[
\Delta \ell(\phi, \phi^*) \lesssim C \left[ \frac{1}{2} \left( \frac{\sqrt{3} F'' T^3}{N_t} \right)^2 + \frac{1}{2} \left( \frac{2N}{N_f} \right)^2 \right]
\]

CC grid is tight!

\[
\max_{\text{all GW chirps}} \{\ell\} \approx \max_{\text{all CCs}} \{\ell\}
\]

search over CCs? the number of CCs is finite!

... but exponentially growing with \( N_t \) (combinatorial)

CC grid is too large to be searched exhaustively!
best CC, step (1): maps to time-frequency

scalar products can be expressed in time or in frequency

Parseval: \[ \int x(t)y^*(t)\,dt = \int X(f)Y^*(f)\,df \]
best CC, step (1): maps to time-frequency

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or in time-frequency

Moyal: \[ \left| \int x(t)y^*(t) \, dt \right|^2 = \iint W_x(t, f)W_y^*(t, f) \, dt \, df \]
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for discrete signals, discrete Wigner-Ville

Moyal: \[ \ell = \frac{1}{2N} \sum_n \sum_m W_x(n, m)W_e(n, m) \]
best CC, step (2): template WV is simple

\[ W_e \text{ is almost Dirac } \approx \delta(m - m_n^{(cc)}) \]

\[
\ell \propto \sum_n \sum_m W_x(n, m) W_e(n, m)
\]

path integral: \( \ell \approx \sum_n W_x(n, m_n^{(cc)}) \)

max_{\phi}\{\ell\} is a longest TF path problem

dynamic programming solves this in polynomial time
**best CC, step (2): template WV is simple**

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*path integral: \( \ell \approx \sum_n W_x(n, m_n^{(cc)}) \)*

\[ \max_{\phi}\{\ell\} \text{ is a longest TF path problem} \]

dynamic programming solves this in polynomial time

which TFR? DWV \hspace{1cm} which pattern search? largest path int. + DP
best CC: check

random CC in Gaussian noise, SNR=20

noise free random CC

discrete Wigner–Ville

random chirp [solid/green] best CC [dashed/red]
**best CC: performance, ROCs (1)**

ROC: detection prob. vs false alarm

![ROC graph](image)

**STS: Signal Track Search**

**TFC: TF Clusters**
best CC: performance, ROCs (2)

“clairvoyant” observer knows incident chirp *a priori*
the SNR of “clairvoyant” observer is set such that ROC fits the other.

reduction factor in the sight distance wrt “clairvoyant” ≈ 2.6
Concluding remarks

• best CC search
• design a template grid which covers the entire set of ("regular") GW chirps
• use original time-frequency scheme to search efficiently through this grid
• robustness comes from the large size of this grid, not from specific property of time-frequency representation
• articles, codes and other resources available at http://www.apc.univ-paris7.fr/~ecm