Outer Boundary Conditions for the Generalized Harmonic Einstein Equations: Stability and Accuracy

Oliver Rinne

Work with the Caltech-Cornell Numerical Relativity Collaboration

Theoretical Astrophysics and Relativity, California Institute of Technology

From Geometry to Numerics, IHP, Paris, November 21, 2006
Outline

1. Introduction
2. Construction of boundary conditions
3. Stability analysis
4. Accuracy comparisons
5. Summary
The initial-boundary value problem

- Consider Einstein’s equations on compact spatial domain $\Omega$ with smooth outer boundary $\partial \Omega$

Boundary conditions should

1. yield a well-posed initial-boundary value problem
2. be compatible with the constraints (*constraint-preserving*)
3. minimize reflections, control incoming gravitational radiation
The initial-boundary value problem

- Consider Einstein’s equations on compact spatial domain $\Omega$ with smooth outer boundary $\partial \Omega$

- Boundary conditions should
  1. yield a well-posed initial-boundary value problem
  2. be compatible with the constraints (constraint-preserving)
  3. minimize reflections, control incoming gravitational radiation
Previous work

- [Friedrich & Nagy 1999] formulation that satisfies all three requirements for the fully nonlinear vacuum Einstein equations (tetrad-based, evolves Weyl tensor)

- Necessary conditions for well-posedness can be verified using pseudo-differential techniques (Fourier-Laplace analysis) [Stewart 1998, Calabrese & Sarbach 2003, Sarbach & Tiglio 2005, Kreiss & Winicour 2006, R 2006]

- Alternate approach to proving well-posedness via semigroup theory [Reula & Sarbach 2005, Nagy & Sarbach 2006]


- Some alternatives: spatial compactification, Cauchy-characteristic and Cauchy-perturbative matching, hyperboloidal slices, ...
Previous work

- [Friedrich & Nagy 1999] formulation that satisfies all three requirements for the fully nonlinear vacuum Einstein equations (tetrad-based, evolves Weyl tensor)

- Necessary conditions for well-posedness can be verified using pseudo-differential techniques (Fourier-Laplace analysis) [Stewart 1998, Calabrese & Sarbach 2003, Sarbach & Tiglio 2005, Kreiss & Winicour 2006, R 2006]

- Alternate approach to proving well-posedness via semigroup theory [Reula & Sarbach 2005, Nagy & Sarbach 2006]


- Some alternatives: spatial compactification, Cauchy-characteristic and Cauchy-perturbative matching, hyperboloidal slices, ...
Previous work

- [Friedrich & Nagy 1999] formulation that satisfies all three requirements for the fully nonlinear vacuum Einstein equations (tetrad-based, evolves Weyl tensor)

- Necessary conditions for well-posedness can be verified using pseudo-differential techniques (Fourier-Laplace analysis) [Stewart 1998, Calabrese & Sarbach 2003, Sarbach & Tiglio 2005, Kreiss & Winicour 2006, R 2006]

- Alternate approach to proving well-posedness via semigroup theory [Reula & Sarbach 2005, Nagy & Sarbach 2006]


- Some alternatives: spatial compactification, Cauchy-characteristic and Cauchy-perturbative matching, hyperboloidal slices, . . .
Previous work

- [Friedrich & Nagy 1999] formulation that satisfies all three requirements for the fully nonlinear vacuum Einstein equations (tetrad-based, evolves Weyl tensor)

- Necessary conditions for well-posedness can be verified using pseudo-differential techniques (Fourier-Laplace analysis) [Stewart 1998, Calabrese & Sarbach 2003, Sarbach & Tiglio 2005, Kreiss & Winicour 2006, R 2006]

- Alternate approach to proving well-posedness via semigroup theory [Reula & Sarbach 2005, Nagy & Sarbach 2006]


- Some alternatives: spatial compactification, Cauchy-characteristic and Cauchy-perturbative matching, hyperboloidal slices, ...
Previous work

- [Friedrich & Nagy 1999] formulation that satisfies all three requirements for the fully nonlinear vacuum Einstein equations (tetrad-based, evolves Weyl tensor)

- Necessary conditions for well-posedness can be verified using pseudo-differential techniques (Fourier-Laplace analysis) [Stewart 1998, Calabrese & Sarbach 2003, Sarbach & Tiglio 2005, Kreiss & Winicour 2006, R 2006]

- Alternate approach to proving well-posedness via semigroup theory [Reula & Sarbach 2005, Nagy & Sarbach 2006]


- Some alternatives: spatial compactification, Cauchy-characteristic and Cauchy-perturbative matching, hyperboloidal slices, . . .
(Generalized) harmonic gauge

- Harmonic coordinates
  \[ \Box x^a = 0 \]

- Principal part of Einstein equations becomes wave operator on metric \( \psi_{ab} \),
  \[ 0 = R_{ab} \simeq -\frac{1}{2} \Box \psi_{ab} \]

- Symmetric hyperbolic system, Cauchy problem is well-posed [Choquet-Bruhat 1952]

- Subject to constraints
  \[ C_a \equiv H_a - \Box x_a = H_a + \Gamma_{ab}^b = 0 \]
(Generalized) harmonic gauge

- **Generalized** harmonic coordinates [Friedrich 1985]

\[ \Box x^a = H^a(x, \psi) \]

- Principal part of Einstein equations becomes wave operator on metric \( \psi_{ab} \),

\[ 0 = R_{ab} \simeq -\frac{1}{2} \Box \psi_{ab} \]

- Symmetric hyperbolic system, Cauchy problem is well-posed [Choquet-Bruhat 1952]

- Subject to constraints

\[ C_a \equiv H_a - \Box x_a = H_a + \Gamma_{ab}^b = 0 \]
(Generalized) harmonic gauge

- **Generalized** harmonic coordinates [Friedrich 1985]
  \[ \Box x^a = H^a(x, \psi) \]

- Principal part of Einstein equations becomes wave operator on metric \( \psi_{ab} \),
  \[ 0 = R_{ab} \simeq - \frac{1}{2} \Box \psi_{ab} \]

- Symmetric hyperbolic system, Cauchy problem is well-posed [Choquet-Bruhat 1952]
- Subject to constraints
  \[ C_a \equiv H_a - \Box x_a = H_a + \Gamma_{ab}^b = 0 \]
(Generalized) harmonic gauge

- **Generalized** harmonic coordinates [Friedrich 1985]
  \[ \Box x^a = H^a(x, \psi) \]

- Principal part of Einstein equations becomes wave operator on metric \( \psi_{ab} \),
  \[ 0 = R_{ab} \approx -\frac{1}{2} \Box \psi_{ab} \]

- Symmetric hyperbolic system, Cauchy problem is well-posed [Choquet-Bruhat 1952]
  - Subject to constraints
  \[ C_a \equiv H_a - \Box x_a = H_a + \Gamma_{ab}^b = 0 \]
(Generalized) harmonic gauge

- **Generalized harmonic coordinates** [Friedrich 1985]

  \[ \Box x^a = H^a(x, \psi) \]

- Principal part of Einstein equations becomes wave operator on metric \( \psi_{ab} \),

  \[ 0 = R_{ab} \simeq -\frac{1}{2} \Box \psi_{ab} \]

- Symmetric hyperbolic system, Cauchy problem is well-posed [Choquet-Bruhat 1952]

- Subject to constraints

  \[ C_a \equiv H_a - \Box x_a = H_a + \Gamma_{ab}^b = 0 \]
First-order reduction

- [Lindblom et al. 2006] Introduce new variables for first time and spatial derivatives of metric

\[ \Pi_{ab} \equiv -t^c \partial_c \psi_{ab}, \quad \Phi_{iab} \equiv \partial_i \psi_{ab} \]

\((t^a \text{ normal to } t = \text{const.} \text{ hypersurfaces, indices } i, j, \ldots = 1, 2, 3)\)

- New constraints

\[ C_{iab} \equiv \partial_i \psi_{ab} - \Phi_{iab} = 0, \quad C_{ijab} \equiv 2\partial_{[i\Phi_j]}ab = 0 \]

- To principal parts, obtain

\[ \partial_t \psi_{ab} \simeq 0, \]
\[ \partial_t \Pi_{ab} \simeq N^k \partial_k \Pi_{ab} - Ng^{ki} \partial_k \Phi_{iab} + \gamma_2 N^k \partial_k \psi_{ab}, \]
\[ \partial_t \Phi_{iab} \simeq N^k \partial_k \Phi_{iab} - N\partial_i \Pi_{ab} + N\gamma_2 \partial_i \psi_{ab}, \]

\((g_{ab} = \psi_{ab} + t_a t_b \text{ spatial metric, } (\partial_t)^a = Nt^a + N^a \text{ lapse & shift})\)
First-order reduction

- [Lindblom et al. 2006] Introduce new variables for first time and spatial derivatives of metric

\[ \Pi_{ab} \equiv -t^c \partial_c \psi_{ab}, \quad \Phi_{iab} \equiv \partial_i \psi_{ab} \]

\((t^a \text{ normal to } t = \text{const. hypersurfaces, indices } i, j, \ldots = 1, 2, 3)\)

- New constraints

\[ C_{iab} \equiv \partial_i \psi_{ab} - \Phi_{iab} = 0, \quad C_{ijab} \equiv 2 \partial_{[i} \Phi_{j]}_{ab} = 0 \]

- To principal parts, obtain

\[ \partial_t \psi_{ab} \simeq 0, \]
\[ \partial_t \Pi_{ab} \simeq N^k \partial_k \Pi_{ab} - N g^{ki} \partial_k \Phi_{iab} + \gamma_2 N^k \partial_k \psi_{ab}, \]
\[ \partial_t \Phi_{iab} \simeq N^k \partial_k \Phi_{iab} - N \partial_i \Pi_{ab} + N \gamma_2 \partial_i \psi_{ab}, \]

\((g_{ab} = \psi_{ab} + t_a t_b \text{ spatial metric, } (\partial_t)^a = N t^a + N^a \text{ lapse & shift})\)
First-order reduction

[\textbf{Lindblom et al. 2006}] Introduce new variables for first time and spatial derivatives of metric

$$\Pi_{ab} \equiv -t^c \partial_c \psi_{ab}, \quad \Phi_{iab} \equiv \partial_i \psi_{ab}$$

($t^a$ normal to $t = \text{const.}$ hypersurfaces, indices $i, j, \ldots = 1, 2, 3$)

New constraints

$$C_{iab} \equiv \partial_i \psi_{ab} - \Phi_{iab} = 0, \quad C_{ijab} \equiv 2\partial_{[i} \Phi_{j]} ab = 0$$

To principal parts, obtain

$$\partial_t \psi_{ab} \simeq 0,$$

$$\partial_t \Pi_{ab} \simeq N^k \partial_k \Pi_{ab} - Ng^{ki} \partial_k \Phi_{iab} + \gamma_2 N^k \partial_k \psi_{ab},$$

$$\partial_t \Phi_{iab} \simeq N^k \partial_k \Phi_{iab} - N\partial_i \Pi_{ab} + N\gamma_2 \partial_i \psi_{ab},$$

($g_{ab} = \psi_{ab} + t_a t_b$ spatial metric, $(\partial_t)^a = N t^a + N^a$ lapse & shift)
Characteristic structure

- System is symmetric hyperbolic, characteristic variables in direction $n_i$ are

$$u_{ab}^{0} = \psi_{ab}, \quad \text{speed 0},$$

$$u_{ab}^{1\pm} = \Pi_{ab} \pm \Phi_{nab} - \gamma 2 \psi_{ab}, \quad \text{speed} - N^n \pm N,$$

$$u_{Aab}^{2} = \Phi_{Aab}, \quad \text{speed} - N^n$$

($v_n \equiv n_i v^i, \; v_A \equiv P_A i v^i, \; \text{boundary metric} \; P_{ij} \equiv g_{ij} - n_i n_j$)

- Note dependence of speeds on normal component $N^n$ of shift
System is symmetric hyperbolic, characteristic variables in direction $n_i$ are

\[
\begin{align*}
  u_{ab}^0 &= \psi_{ab}, & \text{speed } 0, \\
  u_{ab}^{1\pm} &= \Pi_{ab} \pm \Phi_{nab} - \gamma 2\psi_{ab}, & \text{speed } - N^n \pm N, \\
  u_{Aab}^2 &= \Phi_{Aab}, & \text{speed } - N^n
\end{align*}
\]

$(v_n \equiv n_i v^i$, $v_A \equiv P_{Ai} v^i$, boundary metric $P_{ij} \equiv g_{ij} - n_i n_j$)

Note dependence of speeds on normal component $N^n$ of shift
Outline

1. Introduction

2. Construction of boundary conditions

3. Stability analysis

4. Accuracy comparisons

5. Summary
Constraints obey subsidiary system

\[ \Box C_a \approx 0 \]

Set incoming modes of this system to zero at the boundary
(in contrast, [Kreiss & Winicour 2006] use \( C_a \neq 0 \))

Obtain conditions on normal derivatives of 4 components of main incoming fields \( u^{1-} \),

\[ P_C^{ab \cd} \partial_n u^{1-}_{cd} \overset{\dagger}{=} (\text{tangential derivatives}) , \]

where \( P_C \) is projection operator with rank 4

If \( N^n > 0 \) then \( u^{2}_{Aab} \) also need boundary conditions, obtained by requiring

\[ C_{nAab} \overset{\dagger}{=} 0 \Rightarrow \partial_n \Phi_{Abc} \overset{\dagger}{=} \partial_A \Phi_{nbc} \]
Constraint-preserving boundary conditions

- Constraints obey **subsidiary system**
  \[ \Box C_a \simeq 0 \]

- Set incoming modes of this system to zero at the boundary (in contrast, [Kreiss & Winicour 2006] use \( C_a \neq 0 \))

- Obtain conditions on *normal derivatives* of 4 components of main incoming fields \( u^1 \),
  \[ P^C_{ab} \partial_n u^{1-}_{cd} = \text{(tangential derivatives)}, \]
  where \( P^C \) is projection operator with rank 4

- If \( N^a \neq 0 \) then \( u^2_{Aab} \) also need boundary conditions, obtained by requiring
  \[ C_{nAab} \neq 0 \Rightarrow \partial_n \Phi_{ABC} = \partial_A \Phi_{nbc} \]
Constraint-preserving boundary conditions

- Constraints obey subsidiary system
  \[ \Box C_a \simeq 0 \]

- Set incoming modes of this system to zero at the boundary (in contrast, [Kreiss & Winicour 2006] use \( C_a \uparrow 0 \))

- Obtain conditions on normal derivatives of 4 components of main incoming fields \( u^{1-} \),
  \[ P^C_{ab} \partial_n u^{1-}_{cd} \uparrow (\text{tangential derivatives}), \]
  where \( P^C \) is projection operator with rank 4

- If \( N^a \uparrow 0 \) then \( u^2_{Aab} \) also need boundary conditions, obtained by requiring
  \[ C_{nAab} \uparrow 0 \Rightarrow \partial_n \Phi_{Abc} \uparrow \partial_A \Phi_{nbc} \]
Constraint-preserving boundary conditions

- Constraints obey subsidiary system
  \[ \square C_a \cong 0 \]

- Set incoming modes of this system to zero at the boundary (in contrast, [Kreiss & Winicour 2006] use \( C_a \neq 0 \))

- Obtain conditions on normal derivatives of 4 components of main incoming fields \( u^{1-} \),
  \[ P_{ab}^{C} \partial_n u_{cd}^{1-} = (\text{tangential derivatives}), \]

  where \( P^C \) is projection operator with rank 4

- If \( N^n > 0 \) then \( u^{2}_{Aab} \) also need boundary conditions, obtained by requiring
  \[ C_{nAab} \neq 0 \Rightarrow \partial_n \Phi_{Abc} \neq \partial_A \Phi_{nbc} \]
Physical boundary conditions

- Incoming gravitational radiation $\Leftrightarrow$ Newman-Penrose scalar
  \[ \psi_0 = C_{abcd} l^a m^b l^c m^d, \]

  \[ \{ l^a = (t^a + n^a)/\sqrt{2}, k^a = (t^a - n^a)/\sqrt{2}, m^a, \bar{m}^a \} \text{ complex null tetrad} \]

- We impose the BC
  \[ \psi_0 = 0 \]

- Rewrite as
  \[ P^P_{ab} cd \partial_n u^1_{cd} = (\text{tangential derivatives}) + h^P_{ab}, \]

where $P^P$ has rank 2 and is orthogonal to $P^C$

- Lowest level in a hierarchy of perfectly absorbing BCs for linearized gravitational waves [Luisa Buchman’s talk]
Physical boundary conditions

- Incoming gravitational radiation ⇔ Newman-Penrose scalar
  \[ \psi_0 = C_{abcd} l^a m^b l^c m^d, \]
  \[ \{ l^a = (t^a + n^a)/\sqrt{2}, k^a = (t^a - n^a)/\sqrt{2}, m^a, \bar{m}^a \} \text{ complex null tetrad} \]
- We impose the BC
  \[ \psi_0 \equiv 0 \]
- Rewrite as
  \[ P^P_{ab} \partial_n u^1_{cd} \equiv (\text{tangential derivatives}) + h^P_{ab}, \]
  where \( P^P \) has rank 2 and is orthogonal to \( P^C \)
- Lowest level in a hierarchy of perfectly absorbing BCs for linearized gravitational waves [Luisa Buchman’s talk]
Physical boundary conditions

- Incoming gravitational radiation $\Leftrightarrow$ Newman-Penrose scalar
  \[ \Psi_0 = C_{abcd} l^a m^b l^c m^d, \]
  \[ \{ l^a = (t^a + n^a)/\sqrt{2}, k^a = (t^a - n^a)/\sqrt{2}, m^a, \overline{m}^a \} \text{ complex null tetrad} \]
- We impose the BC
  \[ \Psi_0 \dot{=} h^P \]
- Rewrite as
  \[ P^P_{ab} \partial_n u^{1-}_{cd} \dot{=} (\text{tangential derivatives}) + h^P_{ab}, \]
  where $P^P$ has rank 2 and is orthogonal to $P^C$
- Lowest level in a hierarchy of perfectly absorbing BCs for linearized gravitational waves [Luisa Buchman’s talk]
Physical boundary conditions

- Incoming gravitational radiation ⇔ Newman-Penrose scalar
  \[ \psi_0 = C_{abcd} l^a m^b |^c m^d, \]

\[ \{ l^a = (t^a + n^a)/\sqrt{2}, k^a = (t^a - n^a)/\sqrt{2}, m^a, \bar{m}^a \} \] complex null tetrad

- We impose the BC
  \[ \psi_0 = h^P \]

- Rewrite as
  \[ P^P_{ab} \partial_n u^{1-}_{cd} = (\text{tangential derivatives}) + h^P_{ab}, \]

where \( P^P \) has rank 2 and is orthogonal to \( P^C \)

- Lowest level in a hierarchy of perfectly absorbing BCs for linearized gravitational waves [Luisa Buchman’s talk]
Physical boundary conditions

- Incoming gravitational radiation $\Leftrightarrow$ Newman-Penrose scalar

$$\psi_0 = C_{abcd} l^a m^b l^c m^d,$$

$$\{ l^a = (t^a + n^a)/\sqrt{2}, k^a = (t^a - n^a)/\sqrt{2}, m^a, \bar{m}^a\}$$ complex null tetrad

- We impose the BC

$$\psi_0 = h^P$$

- Rewrite as

$$P^P_{ab} \partial_n u^{1-}_{cd} = (\text{tangential derivatives}) + h^P_{ab},$$

where $P^P$ has rank 2 and is orthogonal to $P^C$.

- Lowest level in a hierarchy of perfectly absorbing BCs for linearized gravitational waves [Luisa Buchman’s talk]
Gauge boundary conditions

- Remaining gauge freedom $x^a \rightarrow x^a + \xi^a$ provided that

$$\square \xi^a = 0$$

- Induced metric change

$$\psi_{ab} \rightarrow \psi_{ab} - 2\partial_a \xi_b$$

- Ideally, impose absorbing BC on $\xi^a$
- To leading order in inverse radius, a suitable BC is

$$P^G_{ab} (u^{-1}_{cd} + \gamma_2 \psi_{ab}) = 0$$

where $P^G$ has rank 4 and $P^C + P^P + P^G = I$
Gauge boundary conditions

- Remaining gauge freedom $x^a \rightarrow x^a + \xi^a$ provided that
  \[ \Box \xi^a = 0 \]

- Induced metric change
  \[ \psi_{ab} \rightarrow \psi_{ab} - 2 \partial(a \xi_b) \]

- Ideally, impose absorbing BC on $\xi^a$

- To leading order in inverse radius, a suitable BC is
  \[ P^G_{ab} (u^1_{cd} + \gamma_2 \psi_{ab}) \parallel 0 \]

  where $P^G$ has rank 4 and $P^C + P^P + P^G = I$
Gauge boundary conditions

- Remaining gauge freedom $x^a \rightarrow x^a + \xi^a$ provided that
  \[
  \Box \xi^a = 0
  \]

- Induced metric change
  \[
  \psi_{ab} \rightarrow \psi_{ab} - 2 \partial{(a\xi_b)}
  \]

- Ideally, impose absorbing BC on $\xi^a$

  To leading order in inverse radius, a suitable BC is
  \[
  P^G_{ab} (u^{1-}_{cd} + \gamma_2 \psi_{ab}) = 0
  \]

  where $P^G$ has rank 4 and $P^C + P^P + P^G = I$
Gauge boundary conditions

- Remaining gauge freedom $x^a \rightarrow x^a + \xi^a$ provided that
  \[ \Box \xi^a = 0 \]

- Induced metric change
  \[ \psi_{ab} \rightarrow \psi_{ab} - 2 \partial (a \xi_b) \]

- Ideally, impose absorbing BC on $\xi^a$

- To leading order in inverse radius, a suitable BC is
  \[ P^G_{cd} (u^{1-}_{cd} + \gamma_2 \psi_{ab}) = 0 \]

  where $P^G$ has rank 4 and $P^C + P^P + P^G = \mathbb{I}$
Gauge boundary conditions

- Remaining gauge freedom \( x^a \to x^a + \xi^a \) provided that
  \[ \Box \xi^a = 0 \]

- Induced metric change
  \[ \psi_{ab} \to \psi_{ab} - 2 \partial_\alpha (a \xi^b) \]

- Ideally, impose absorbing BC on \( \xi^a \)

- To leading order in inverse radius, a suitable BC is
  \[ P^G_{cd} (u^{1-}_{cd} + \gamma_2 \psi_{ab}) = h^G_{cd} \]

  where \( P^G \) has rank 4 and \( P^C + P^P + P^G = I \)
Outline

1. Introduction
2. Construction of boundary conditions
3. Stability analysis
4. Accuracy comparisons
5. Summary

Oliver Rinne (Caltech)  GH Boundary Conditions: Stability & Accuracy  GeoNum 11/21/2006  13 / 33
Fourier-Laplace analysis

- Consider high-frequency perturbations about any given spacetime
- Obtain linear symmetric hyperbolic system with constant coefficients
- Solve by Laplace transform in time and Fourier transform in space
- Boundary conditions imply linear system of equations for integration constants
- Study zeros of its (complex) determinant $\Rightarrow$ necessary conditions for well-posedness (determinant condition and Kreiss condition)
- GH system satisfies both conditions [R 2006]
Consider high-frequency perturbations about any given spacetime
Obtain linear symmetric hyperbolic system with constant coefficients
Solve by Laplace transform in time and Fourier transform in space
Boundary conditions imply linear system of equations for integration constants
Study zeros of its (complex) determinant \(\Rightarrow\) necessary conditions for well-posedness (determinant condition and Kreiss condition)
GH system satisfies both conditions \([R\ 2006]\)
Fourier-Laplace analysis

- Consider high-frequency perturbations about any given spacetime
- Obtain linear symmetric hyperbolic system with constant coefficients
- Solve by Laplace transform in time and Fourier transform in space
- Boundary conditions imply linear system of equations for integration constants
- Study zeros of its (complex) determinant \(\Rightarrow\) necessary conditions for well-posedness (determinant condition and Kreiss condition)
- GH system satisfies both conditions [R 2006]
Consider high-frequency perturbations about any given spacetime

Obtain linear symmetric hyperbolic system with constant coefficients

Solve by Laplace transform in time and Fourier transform in space

Boundary conditions imply linear system of equations for integration constants

Study zeros of its (complex) determinant $\Rightarrow$ necessary conditions for well-posedness (determinant condition and Kreiss condition)

GH system satisfies both conditions [R 2006]
Fourier-Laplace analysis

- Consider high-frequency perturbations about any given spacetime
- Obtain linear symmetric hyperbolic system with constant coefficients
- Solve by Laplace transform in time and Fourier transform in space
- Boundary conditions imply linear system of equations for integration constants
- Study zeros of its (complex) determinant \(\implies\) necessary conditions for well-posedness (determinant condition and Kreiss condition)
- GH system satisfies both conditions [R 2006]
Fourier-Laplace analysis

- Consider high-frequency perturbations about any given spacetime
- Obtain linear symmetric hyperbolic system with constant coefficients
- Solve by Laplace transform in time and Fourier transform in space
- Boundary conditions imply linear system of equations for integration constants
- Study zeros of its (complex) determinant ⇒ necessary conditions for well-posedness (determinant condition and Kreiss condition)
- GH system satisfies both conditions [R 2006]
Towards sufficient conditions

- Kreiss condition implies that solution can be estimated in terms of boundary data \textit{(boundary-stable)}
- One would also like to control
  - source terms \textit{(well-posedness in the generalized sense)}
  - initial data \textit{(well-posedness)}
- Proof via symmetrizer construction \cite{Kreiss 1970}
- Technique not applicable to boundary conditions of \textit{differential} type
- Can show that system is free of \textit{weak instabilities} with polynomial time dependence
Towards sufficient conditions

- Kreiss condition implies that solution can be estimated in terms of boundary data (boundary-stable)
- One would also like to control
  - source terms (well-posedness in the generalized sense)
  - initial data (well-posedness)
- Proof via symmetrizer construction [Kreiss 1970]
- Technique not applicable to boundary conditions of differential type
- Can show that system is free of weak instabilities with polynomial time dependence
Towards sufficient conditions

- Kreiss condition implies that solution can be estimated in terms of boundary data (boundary-stable)
- One would also like to control
  - source terms (well-posedness in the generalized sense)
  - initial data (well-posedness)
- Proof via symmetrizer construction [Kreiss 1970]
  - Technique not applicable to boundary conditions of differential type
  - Can show that system is free of weak instabilities with polynomial time dependence
Towards sufficient conditions

- Kreiss condition implies that solution can be estimated in terms of boundary data (boundary-stable)
- One would also like to control
  - source terms (well-posedness in the generalized sense)
  - initial data (well-posedness)
- Proof via symmetrizer construction [Kreiss 1970]
- Technique not applicable to boundary conditions of differential type
- Can show that system is free of weak instabilities with polynomial time dependence
Towards sufficient conditions

- Kreiss condition implies that solution can be estimated in terms of boundary data (boundary-stable)
- One would also like to control
  - source terms (well-posedness in the generalized sense)
  - initial data (well-posedness)
- Proof via symmetrizer construction [Kreiss 1970]
- Technique not applicable to boundary conditions of differential type
- Can show that system is free of weak instabilities with polynomial time dependence
Numerical robust stability test

- Consider fixed background solution (Minkowski or Schwarzschild)
- Add small random perturbations to initial data, boundary data and right-hand-sides of evolution equations
- Evolve on domain $T^2 \times \mathbb{R}$, impose BCs in transverse direction
- Pseudospectral collocation method [Caltech-Cornell Spectral Einstein Code]
- Monitor error (deviation from background solution) and constraint violations
Consider fixed background solution (Minkowski or Schwarzschild)

Add small random perturbations to \textit{initial data, boundary data and right-hand-sides of evolution equations}

Evolve on domain $T^2 \times \mathbb{R}$, impose BCs in transverse direction

Pseudospectral collocation method [Caltech-Cornell Spectral Einstein Code]

Monitor error (deviation from background solution) and constraint violations
Numerical robust stability test

- Consider fixed background solution (Minkowski or Schwarzschild)
- Add small random perturbations to initial data, boundary data and right-hand-sides of evolution equations
- Evolve on domain $T^2 \times \mathbb{R}$, impose BCs in transverse direction
- Pseudospectral collocation method [Caltech-Cornell Spectral Einstein Code]
- Monitor error (deviation from background solution) and constraint violations
Consider fixed background solution (Minkowski or Schwarzschild)

Add small random perturbations to *initial data, boundary data and right-hand-sides of evolution equations*

Evolve on domain $T^2 \times \mathbb{R}$, impose BCs in transverse direction

Pseudospectral collocation method [Caltech-Cornell Spectral Einstein Code]

Monitor error (deviation from background solution) and constraint violations
Numerical robust stability test

- Consider fixed background solution (Minkowski or Schwarzschild)
- Add small random perturbations to *initial data, boundary data and right-hand-sides of evolution equations*
- Evolve on domain $T^2 \times \mathbb{R}$, impose BCs in transverse direction
- Pseudospectral collocation method [Caltech-Cornell Spectral Einstein Code]
- Monitor error (deviation from background solution) and constraint violations
Flat space without shift

\[ N^i = (0, 0, 0) \]

Random data amplitude \(10^{-10}\)
Flat space with constant shift

\[ N^i = (0.5, 0.5, 0) \]

Random data amplitude \(10^{-10}\)
Random data amplitude $10^{-6}$
Outline

1. Introduction
2. Construction of boundary conditions
3. Stability analysis
4. Accuracy comparisons
5. Summary
Some alternative boundary treatments

- Freezing all the incoming fields

\[ u_{ab}^1 = 0 \quad \text{(and)} \quad u_{Aab}^2 = 0 \quad \text{if} \quad N^n > 0 \]

- Sommerfeld boundary conditions (popular for BSSN formulation), for spherical boundary of radius \( r = R \),

\[ (\partial_t + \partial_r + \frac{1}{R}) \psi_{ab} = 0 \]

- Spatial compactification [Pretorius 2005]
  - Choose mapping \( r \rightarrow x(r) \) that maps spatial infinity to a finite coordinate location, e.g. \( x = \arctan r \)
  - Discretize uniformly in \( x \)
  - Apply low-pass frequency filter to damp waves as they travel out
Some alternative boundary treatments

- Freezing all the incoming fields
  \[ u^1_{ab} \dot{=} 0 \quad \text{(and)} \quad u^2_{Aab} \dot{=} 0 \quad \text{if} \quad N^n \dot{>} 0 \]

- Sommerfeld boundary conditions (popular for BSSN formulation), for spherical boundary of radius \( r = R \),
  \[ (\partial_t + \partial_r + \frac{1}{R}) \psi_{ab} \dot{=} 0 \]

- Spatial compactification [Pretorius 2005]
  - Choose mapping \( r \rightarrow x(r) \) that maps spatial infinity to a finite coordinate location, e.g. \( x = \arctan r \)
  - Discretize uniformly in \( x \)
  - Apply low-pass frequency filter to damp waves as they travel out
Some alternative boundary treatments

- Freezing all the incoming fields

\[ u_{ab}^{1-} = 0 \quad (\text{and} \quad u_{Aab}^{2} = 0 \quad \text{if} \quad N^{n} > 0) \]

- Sommerfeld boundary conditions (popular for BSSN formulation), for spherical boundary of radius \( r = R \),

\[ (\partial_t + \partial_r + \frac{1}{R})\psi_{ab} = 0 \]

- Spatial compactification [Pretorius 2005]
  - Choose mapping \( r \rightarrow x(r) \) that maps spatial infinity to a finite coordinate location, e.g. \( x = \arctan r \)
  - Discretize uniformly in \( x \)
  - Apply low-pass frequency filter to damp waves as they travel out
Our test problem

[Ongoing work with Lee Lindblom and Mark Scheel]

- **Background solution:** Schwarzschild black hole (mass $M = 1$)
- Add outgoing quadrupole wave perturbation [Teukolsky 1982], amplitude $4 \times 10^{-3}$ (odd-parity)
- Evolve on a spherical shell extending from $r = 1.9$ (just inside the horizon) out to
  - $R = 1000$ (*reference solution*)
  - $R = 41.9, 81.9, \ldots$
- On the smaller domain, either impose the boundary conditions described in this talk or apply one of the alternative methods
- Compute difference of the two numerical solutions, compare in- and outgoing radiation ($\psi_0$ and $\psi_4$), \ldots
Our test problem

[Ongoing work with Lee Lindblom and Mark Scheel]

- Background solution: Schwarzschild black hole (mass $M = 1$)
- Add outgoing quadrupole wave perturbation [Teukolsky 1982], amplitude $4 \times 10^{-3}$ (odd-parity)
- Evolve on a spherical shell extending from $r = 1.9$ (just inside the horizon) out to
  - $R = 1000$ (reference solution)
  - $R = 41.9, 81.9, \ldots$

- On the smaller domain, either impose the boundary conditions described in this talk or apply one of the alternative methods
- Compute difference of the two numerical solutions, compare in- and outgoing radiation ($\Psi_0$ and $\Psi_4$), \ldots
Our test problem

[Ongoing work with Lee Lindblom and Mark Scheel]

- Background solution: Schwarzschild black hole (mass $M = 1$)
- Add outgoing quadrupole wave perturbation [Teukolsky 1982], amplitude $4 \times 10^{-3}$ (odd-parity)
- Evolve on a spherical shell extending from $r = 1.9$ (just inside the horizon) out to
  - $R = 1000$ (reference solution)
  - $R = 41.9, 81.9, \ldots$

- On the smaller domain, either impose the boundary conditions described in this talk or apply one of the alternative methods
- Compute difference of the two numerical solutions, compare in- and outgoing radiation ($\Psi_0$ and $\Psi_4$), \ldots
Our test problem

[Ongoing work with Lee Lindblom and Mark Scheel]

- Background solution: Schwarzschild black hole (mass $M = 1$)
- Add outgoing quadrupole wave perturbation [Teukolsky 1982], amplitude $4 \times 10^{-3}$ (odd-parity)
- Evolve on a spherical shell extending from $r = 1.9$ (just inside the horizon) out to
  - $R = 1000$ (reference solution)
  - $R = 41.9, 81.9, \ldots$

- On the smaller domain, either impose the boundary conditions described in this talk or apply one of the alternative methods
- Compute difference of the two numerical solutions, compare in- and outgoing radiation ($\psi_0$ and $\psi_4$), \ldots
Our test problem

[Ongoing work with Lee Lindblom and Mark Scheel]

- Background solution: Schwarzschild black hole (mass $M = 1$)
- Add outgoing quadrupole wave perturbation [Teukolsky 1982], amplitude $4 \times 10^{-3}$ (odd-parity)
- Evolve on a spherical shell extending from $r = 1.9$ (just inside the horizon) out to
  - $R = 1000$ (reference solution)
  - $R = 41.9, 81.9, \ldots$

- On the smaller domain, either impose the boundary conditions described in this talk or apply one of the alternative methods
- Compute difference of the two numerical solutions, compare in- and outgoing radiation ($\psi_0$ and $\psi_4$), \ldots
Old * (solid) vs. new (dotted) CPBCs
(* without the $\gamma_2\psi$ term in the gauge BCs)

$R = 41.9, (N_r, L) = (21, 8), (31, 10), (41, 12), (51, 41)$
Freezing (solid) vs. new CP (dotted) BCs

\[ R = 41.9, \ (N_r, L) = (21, 8), (31, 10), (41, 12), (51, 41) \]
Sommerfeld (solid) vs. new CP (dotted) BCs

\[ R = 41.9, \ (N_r, L) = (21, 8), (31, 10), (41, 12), (51, 41) \]
Tan compactification with various filters vs. new CPBCs

New constraint-preserving BCs
- Kreiss-Oliger filter ($\epsilon = 1$) applied to RHS
- Hesthavenn filter ($\sigma = 0.76, \rho = 13$) applied to RHS
- Kreiss-Oliger filter ($\epsilon = 0.25$) applied to solution
- Hesthavenn filter ($\sigma = 0.76, \rho = 13$) applied to solution

$$R = 41.9, \ (N_r, L) = (51, 14)$$
Tan compactification with best filter (solid) vs. new CPBCs (dotted)

Hesthavven filter applied to solution,

\[ R = 41.9, \ (N_r, L) = (21, 8), (31, 10), (41, 12), (51, 41) \]
Accuracy of extracted $\Psi_4$

$R = 41.9, (N_r, L) = (31, 10), (51, 41)$
The reflection coefficient: theory vs. “experiment”

[Buchman & Sarbach 2006] predict for our CPBCs

\[ \frac{\psi_0}{\psi_4} = \frac{4}{9} (kR)^{-4} + O[(kR)^{-5}] \]

\( R = 41.9 \)  

\( R = 121.9 \)

\( (N_r, L) = (51, 14) \)
Outline

1. Introduction
2. Construction of boundary conditions
3. Stability analysis
4. Accuracy comparisons
5. Summary
Summary

- Constructed a set of constraint-preserving and radiation-controlling boundary conditions for the generalized harmonic Einstein equations.
- Verified necessary conditions for well-posedness using the Fourier-Laplace technique, supported by numerical robust stability tests.
- Numerical results indicate that our BCs cause significantly less reflections than alternate methods such as spatial compactification or Sommerfeld BCs.
Summary

- Constructed a set of constraint-preserving and radiation-controlling boundary conditions for the generalized harmonic Einstein equations.
- Verified necessary conditions for well-posedness using the Fourier-Laplace technique, supported by numerical robust stability tests.
- Numerical results indicate that our BCs cause significantly less reflections than alternate methods such as spatial compactification or Sommerfeld BCs.
Summary

- Constructed a set of constraint-preserving and radiation-controlling boundary conditions for the generalized harmonic Einstein equations.
- Verified necessary conditions for well-posedness using the Fourier-Laplace technique, supported by numerical robust stability tests.
- Numerical results indicate that our BCs cause significantly less reflections than alternate methods such as spatial compactification or Sommerfeld BCs.
<table>
<thead>
<tr>
<th>Reference</th>
<th>Authors</th>
<th>Title</th>
<th>Journal</th>
<th>Volume/Issue</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>[R 2006]</td>
<td>O. Rinne</td>
<td>Stable radiation-controlling boundary conditions for the generalized harmonic Einstein equations</td>
<td><em>Class. Quantum Grav.</em> <strong>23</strong>(22) 6275–6300</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[Kreiss &amp; Winicour 2006]</td>
<td>H.-O. Kreiss and J. Winicour</td>
<td>Problems which are well-posed in a generalized sense with applications to the Einstein equations</td>
<td><em>Class. Quantum Grav.</em> <strong>23</strong>(16) S405–S420</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Spatial compactification: details

Compactification map e.g.

- \( r(x) = R \tan(\pi x / 4R) \)
  
  *(Tan mapping)*

- \( r(x) = \begin{cases} 
  x, & 0 \leq x < R \\
  R^2 / (2R - x), & R \leq x < 2R 
\end{cases} \)
  
  *(Inverse mapping)*

Filter function e.g.

- \( f(k) = 1 - \epsilon \sin^4(\pi k / 2k_{\text{max}}) \),
  
  where \( 0 \leq \epsilon \leq 1 \)
  
  *(Kreiss-Oliger filter)*

- \( f(k) = \exp[-(k / \sigma k_{\text{max}})^p] \),
  
  typically \( \sigma = 0.76, \ p = 13 \)
  
  *(Hesthaven filter)*