Binary black holes with helical Killing vector

C. Klein, Leipzig, MPI for Mathematics in the Sciences.
C.K., O. Richter, Ernst Equation and Riemann Surfaces, LNP 685 (Springer) (2005)
Outline

- Introduction
- Quasi-stationary approximation
- Projection formalism
- 2+1-decomposition
- Outlook
Introduction

- coalescing binary black holes strongest sources for gravitational radiation
- difficult relativistic problem (no symmetries), pure vacuum
- realistic initial values?
- Schild: quasi-circular orbits for two charged bodies in electrodynamics (incoming radiation)
- helical Killing vector, asymptotically $\partial_t + \Omega \partial \phi$
Charges in Maxwell theory

- charges sources of the Maxwell equations, four-dimensional notation, Lorentz gauge, linear equations,

\[ \Box A^\mu = j^\mu \]

explicit expressions for the potentials in terms of retarded integrals over the charges

- charges move according to Lorentz force, therefore non-linear problem (radiation back-reaction)

- Schild: stationary approximation for the quasi-circular motion, incoming radiation compensates outgoing radiation (sequence of circular orbits)
Binary black holes

- pure vacuum problem, no symmetries
- 3+1 decomposition
  \[ ds^2 = -N^2 dt^2 + \gamma_{ab}(dx^a + \beta^a dt)(dx^b + \beta^b dt) \]
- 10 Einstein equations \( R_{\mu\nu} = 0 \) split in 6 time evolution and 4 constraint equations
- constraints constrain the initial values \( \gamma_{ab}(t_0), \gamma_{ab,t}(t_0) \), 4 equations plus 4 gauge freedoms for 12 quantities, underdetermined system, problem: not known how to encode physics in the initial data (gravitational radiation included?)
What has been done?

- post-Newtonian calculations: calculations to order 3pN, resummation of the perturbation series (Blanchet, Buonanno, Damour, Schäfer,...).

- initial data for binary black holes: numerical solution of the Lichnerowicz equations for Bowen-York initial data (conformally flat spatial metric), helical KV (Baumgarte, Cook, Shapiro,...)

- IWM spacetimes (Meudon): toy model for gravitation, theory without radiation: $\gamma_{ab}$ conformally flat, binary IWM black holes with helical KV (non-regular horizon, Kerr not included).
Helical Killing vector

- asymptotically $\partial_t' + \Omega \partial_\phi$, choose $\xi = \partial_t$

- quotient space metric (Ehlers, Geroch)

$$ds^2 = -f(dt + k_a dx^a)(dt + k_b dx^b) + \frac{1}{f} h_{ab} dx^a dx^b$$

$f$: norm of the Killing vector, $\xi_a = -f(1, k_a)$

- Maxwell-type equation

$$\frac{1}{2} D_a (f^2 k^{ab}) = 0, \quad k_{ab} = k_{a,b} - k_{b,a}$$
• twist potential \((h = \det(h_{ab}))\)

\[ k^{ab} = \frac{1}{\sqrt{h} f^2} \epsilon^{abc} \partial_c b \]

• Ernst potential \(\mathcal{E} = f + ib\)

\[ f D_a D^a \mathcal{E} = \frac{f}{\sqrt{h}} (\sqrt{h} h^{ab} \mathcal{E}_a)_b = D_a \mathcal{E} D^a \mathcal{E} \]

corresponds to the 4 constraint equations

• non-linear sigma model

\[ R^{(3)}_{ab} = \frac{1}{2f^2} \Re(\mathcal{E}_a \bar{\mathcal{E}}_b) \]

• 3-dimensional gravity with sigma model ’matter’ determined by the Ernst equation
Projection formalism

- advantage: less and simpler equations
- drawback: singular equations for $f=0$ ($f$ changes sign at the light cylinder, numerical problems?)
Minkowski in rotating coordinates

- Minkowski: \( f = 1, b = 0, \)
  rotating coordinates \( \phi' = \phi - \Omega t \)

\[
f' = 1 - \Omega^2 \rho^2, \quad b' = 2\Omega z
\]

- \( \rho < 1/\Omega: \ f > 0, \)
  \( \rho > 1/\Omega: \ f < 0, \)
  \( \rho = 1/\Omega: \) light cylinder (observer rotates with \( c \))

- transformed metric \( h_{ab} \) (rescaled with \( f \)), \( h_{\phi\phi} \) invariant, rest

\[
h'_{ab} = (1 - \Omega^2 \rho^2) h_{ab}
\]
• signature change from $+3$ to $-1$ at the light cylinder. No signature change of 4d metric, but $t$ and $\phi$ change roles.

• Ernst equation in non-rotating coordinates

$$f \Delta \mathcal{E} = (\nabla \mathcal{E})^2$$

• in rotating coordinates, Laplace operator replaced by $\mathcal{L}$

$$\mathcal{L} \mathcal{E} = \mathcal{E}_{\rho \rho} + \frac{1}{\rho} \mathcal{E}_\rho + \mathcal{E}_{zz} + \left(\frac{1}{\rho^2} - \Omega^2\right) \mathcal{E}_{\phi \phi}$$

elliptic equation inside the light cylinder, hyperbolic outside (if $\phi$-dependent): symmetric positive system, unique solution (Torre), numerical studies (Whelan et al.)
Horizons, 2+1 decomposition

- Killing horizon \((f = 0)\) gives local concept
- 2 + 1 decomposition: foliation by spheres
  \[
  h_{ab} dx^a dx^b = s_{\alpha\beta} (dx^\alpha + B^\alpha dr)(dx^\beta + B^\beta dr) + A^2 dr^2
  \]
- 3 parabolic equations (‘constraint’), 3 elliptic equations
- singularities: horizon, light cylinder, infinity
2+1 decomposition

\[ r, \theta, \phi \]
• horizon: regular singularity (expansion in $t = r - R$ with $\theta, \phi$ dependent coefficients)

$$f \sim A^2 \sim (r - R)^2,$$

locally like Kerr black hole

• infinity not regular, formal expansion

$$y = \sum_{n=1}^{\infty} \frac{y_n(r, \theta, \phi)}{r^n}$$

$r$-dependence of $y_n$ oscillatory terms

• light cylinder: regular singularity at surface with cylindrical topology, precise location unknown in coordinate system adapted to horizon ($f \sim A^2$)
Outlook

- analytical task: global existence, asymptotic behavior
- numerical task: multi-domain spectral methods (Lorene)
- physical task: Killing vector approximate, only valid in finite region, matching to asymptotically flat spacetime
Bispherical coordinates

\[
\begin{align*}
x &= \frac{a \sin \theta \cos \psi}{\cosh \eta - \cos \theta}, \\
y &= \frac{a \sin \theta \sin \psi}{\cosh \eta - \cos \theta}, \\
z &= \frac{a \sinh \eta}{\cosh \eta - \cos \theta},
\end{align*}
\]

Laplace equation \( \Delta F = 0 \)

\[
F(\eta, \theta, \psi) = \sqrt{\cosh \eta - \cos \theta} \sum_{l,m} H_l(\eta) Y_{lm}(\theta, \psi)
\]

\[
H_l(\eta) = a_l e^{(l+\frac{1}{2})\eta} + b_l e^{-(l+\frac{1}{2})\eta}
\]