Higher-Order Corrections in String & M-theory

and

Generalised Holonomy

High Energy, Cosmology and Strings

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Plan

- Higher-Order Corrections in String Theory
- Deformations of Special-Holonomy Backgrounds
- Preservation of Supersymmetry
- Higher-Order Corrections in M-Theory
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Gravity from String Theory

One of the miracles of string theory is that it embodies general covariance, and gravity, albeit in an *a priori* rather non-transparent way.

It shows up even in perturbative string calulations around a flat Minkowski spacetime background. The 3-graviton scattering amplitude in string theory is consistent with the 3-point interaction implied by tree-level scattering in Einstein gravity.

The 4-graviton string scattering amplitude has a contribution that is also consistent with the Einstein-Hilbert term. However, there is an additional string term that is not explained by Einstein-Hilbert gravity. It is in fact the first indication of a higher-order correction to Einstein gravity:

$$I = \int d^{10x} \sqrt{-g} \Big[R + c\, \alpha'^3\, (\text{Riemann})^4 + \cdots \Big]$$

where $(\text{Riemann})^4$ is quartic in curvature.

Corrections to Gravity Backgounds in String Theory

Not only gravity, but the entire leading-order effective supergravity action receives higher-order string corrections. The detailed forms of some of these corrections, even at the 4-point level, are not known. However, if we restrict attention to the gravity (and dilaton) sector, then all the corrections in the effective action up to order ${\alpha'}^3$ are known. This allows, in particular, a detailed discussion of the ${\alpha'}^3$ corrections to purely gravitational backgrounds which, at leading order, satisfied the vacuum Einstein equations.

One particularly interesting question concerns the fate of leading-order gravitational backgrounds with special holonomy, since these, at leading order, are supersymmetric. Examples are

$$(\text{Minkowski})_4 \times K_6\,, \quad (\text{Minkowski})_3 \times K_7\,, \quad (\text{Minkowski})_2 \times K_8$$

where K_6 is a Ricci-flat Calabi-Yau 6-manifold, K_7 is a 7-manifold of G_2 holonomy, and K_8 is an 8-dimensional Ricci-flat Calabi-Yau manifold, a hyper-Kähler manifold or a manifold of Spin(7) holonomy.

Tree-Level Corrections to Type IIA or IIB Strings

In the gravity/dilaton sector, the corrected effective action up to order α'^3 is given by

$$\mathcal{L} = \sqrt{-g}\, e^{-2\phi} \Big(R + 4(\partial\phi)^2 - c\,\alpha'^3\, Y \Big)$$

where c is a known pure-number constant (proportional to $\zeta(3)$). Y is quartic in curvature. The equations of motion are

$$\begin{aligned} R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi &= c\,\alpha'^3\, X_{\mu\nu} \\ \nabla^2\phi - 2(\partial\phi)^2 &= \tfrac{1}{2} c\,\alpha'^3\, (Y - g^{\mu\nu}\, X_{\mu\nu}) \end{aligned}$$

where

$$X_{\mu\nu} = \frac{e^{2\phi}}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^{10}x\, \sqrt{-g}\, e^{-2\phi}\, Y$$

The quartic curvature correction Y is quite complicated, as a ten-dimensional Riemannian expression. With care, we can use a simpler eight-dimensionally covariant light-cone expression, for the special case of (Minkowski)$\times K$ backgrounds.

The Quartic-Curvature Correction

The quartic curvature invariant is given, in light-cone gauge, by

$$Y \propto (t^{i_1\cdots i_8}\, t^{j_1\cdots j_8} - \tfrac{1}{4}\epsilon^{i_1\cdots i_8}\, \epsilon^{j_1\cdots j_8})\, R_{i_1 i_2 j_1 j_2} \cdots R_{i_7 i_8 j_7 j_8}$$

and $t^{i_1\cdots i_8}$ is defined by

$$t^{i_1\cdots i_8}\, M_{i_1 i_2} \cdots M_{i_7 i_8} = 24\mathrm{tr} M^4 - 6(\mathrm{tr} M^2)^2\,, \quad \text{for all } M_{ij} = -M_{ji}$$

It was shown by Gross and Witten that Y could be written as a Berezin integral over $SO(8)$ Majorana spinors $\psi = (\psi_+, \psi_-)$:

$$Y \propto \int d^{16}\psi\ \exp\left[(\bar\psi_+ \Gamma^{ij} \psi_+)(\bar\psi_- \Gamma^{k\ell} \psi_-)\, R_{ijk\ell}\right]$$

Since the integrability condition for a covariantly-constant spinor η in the transverse 8-space is $[\nabla_i, \nabla_j]\eta = \frac{1}{4} R_{ijk\ell}\, \Gamma^{k\ell} \eta = 0$, it follows that a leading-order supersymmetric background will have a spinor zero-mode for at least one of the right-handed or left-handed spinors in the Berezin integral, and hence $Y = 0$.

Corrections to (Minkowski)$_4 \times K_6$

Corrections to Ricci-flat Calabi-Yau manifolds were analysed long ago. It was shown (Freeman & Pope, 1986) that the variation of Y, calculated from the Berezin integral, gives

$$X_{ij} = \nabla_{\hat{i}} \nabla_{\hat{j}} S \,, \qquad \text{where for any } V_i \,,\ V_{\hat{i}} \equiv J_i{}^j \, V_j$$

J is the Kähler form of the original CY background metric, and

$$S = R_{ijk\ell} \, R^{k\ell mn} \, R_{mn}{}^{ij} - 2 R_{ijk\ell} \, R^{km\ell n} \, R_m{}^i{}_n{}^j$$

is the 6-dimensional Euler density. (This agrees with sigma-model beta function calculations by Grisaru et al.)

The corrected equations of motion then imply:

$$R_{ij} = c \, \alpha'^3 \, (\nabla_i \nabla_j + \nabla_{\hat{i}} \nabla_{\hat{j}}) S \,, \qquad \phi = -\tfrac{1}{2} c \, \alpha'^3 \, S$$

(Quantities on RHS calculated using the leading-order background; corrections are valid to order α'^3.) In complex coordinates this corrected Einstein equation is $R_{\alpha\bar{\beta}} = c \, \alpha'^3 \, \partial_\alpha \partial_{\bar{\beta}} S$. The first Chern class still vanishes, but $SU(3) \to U(3)$ holonomy. What happens to supersymmetry?

Supersymmetry in Corrected $(\text{Minkowski})_4 \times K_6$

The leading-order supersymmetry transformation rules also receive α'^3 corrections; their detailed form has recently been obtained (Peeters, Vanhove, Westerberg). There is a general expectation that supersymmetry should survive the corrections. This was studied by Candelas, Freeman, Pope, Sohnius & Stelle (1986) for the 6-dimensional Calabi-Yau case: Can we at least conjecture an α'^3 correction that will make this happen?

The modification of $\delta\psi_\mu = \nabla_\mu \epsilon$ to $\delta\psi_\mu = D_\mu \epsilon$, where

$$D_i = \nabla_i + \frac{i}{2} c\, \alpha'^3 \, (\nabla_{\hat{i}} S)$$

has as integrability condition precisely the corrected Einstein equation in the CY background. In the corrected backgound, we shall have Killing spinors satisfying the corrected condition $D_i \, \eta = 0$; hence supersymmetry.

We can propose $\delta\psi_\mu = D_\mu \epsilon$ as the corrected SUSY transformation in the CY background, but since it involves the explicit use of the Kähler form (hidden in the hat), we must make sure that it is also expressible as a fully Riemannian expression, which specialises to D_i in CY backgrounds.

Riemannian Form of Supersymmetry Correction

A Killing spinor in the leading-order CY background satisfies $\Gamma_{j}\eta = -i\,\Gamma_{\hat{j}}\eta$. This allows us to write a Riemannian expression that reduces to $D_i = \nabla_i + \frac{i}{2} c\,\alpha'^3\,(\nabla_{\hat{i}} S)$ in a six-dimensional CY background (CFPSS):

$$D_i = \nabla_i + \tfrac{3}{4} c\,\alpha'^3\,\nabla_s R^r{}_{ik\ell}\,R^s{}_{tmn}\,R^t{}_{rpq}\,\Gamma^{k\ell mnpq}$$

An alternative form, obtained by dualising in the transverse 8-space, is

$$D_i = \nabla_i - 6c\,\alpha'^3\,\nabla_s R_{ipk\ell}\,R^{st\ell n}\,R_t{}^p{}_n{}^q\,\Gamma_{qk}$$

These, extended to the full index range, provide candidate ten-dimensional Riemannian expressions for the α'^3 correction to the gravitino transformation rule in string theory, that would satisfy the desideratum of implying that the supersymmetry of leading-order $(\text{Minkowski})_4 \times K_6$ backgrounds is preserved in the face of string corrections at order α'^3.

What about leading-order $(\text{Minkowski})_3 \times K_7$ or $(\text{Minkowski})_2 \times K_8$ backgrounds? Will these remain supersymmetric? What is D_i for these?

Corrections to G_2 Holonomy $(\text{Minkowski})_3 \times K_7$

We can view these as $(\text{Minkowski})_2 \times K_8$, where $K_8 = R \times K_7$. With K_7 having G_2 holonomy, we shall have one covariantly-constant $SO(8)$ spinor zero-mode of each chirality. The Berezin integral for Y again vanishes in the background, and its variation can be nicely expressed in terms of special structures on the G_2 manifold (Lü, Pope, Stelle, Townsend):

$$X_{ij} = c_{ikm}\, c_{j\ell n}\, \nabla^k \nabla^\ell\, Z^{mn}$$

where $c_{ijk} = i\, \bar\eta \Gamma_{ijk} \eta$ is the associative 3-form and

$$Z^{mn} \equiv \tfrac{1}{32} \epsilon^{m i_1 \cdots i_6} \epsilon^{n j_1 \cdots j_6}\, R_{i_1 i_2 j_1 j_2} \cdots R_{i_5 i_6 j_5 j_6}$$

From the corrected string equations, we find that on K_7 we now have

$$R_{ij} = c\alpha'^3 \, (\nabla_i \nabla_j S + c_{ikm}\, c_{j\ell n}\, \nabla^k \nabla^\ell\, Z^{mn})\,, \qquad \phi = -\tfrac{1}{2} c\alpha'^3\, S$$

where $S = g_{ij}\, Z^{ij}$ is the 6-dimensional Euler integrand again. Since G_2 manifolds are Ricci-flat, the correction here has destroyed the special holonomy completely. But, in a generalised sense, maybe it hasn't...

Supersymmetry in Corrected $(\text{Minkowski})_3 \times K_7$

Can we again modify the supersymmetry transformation rule in such a way that the corrected G_2 background will again remain supersymmetric? We can again ask for a modification of the gravitino transformation rule, to $\delta\psi_\mu = D_\mu\,\epsilon$, where $D_i = \nabla_i + c\,\alpha'^3\,Q_i$, and require that the integrability condition $[D_i, D_j]\epsilon = 0$ give the corrected G_2 Einstein equation to order α'^3. We find

$$D_i = \nabla_i - \frac{i}{2} c\,\alpha'^3\,c_{ijk}\,(\nabla^j Z^{k\ell})\,\Gamma_\ell$$

This, and the corrected G_2 Einstein equation, both reduce to the previous CY results if we take $K_7 = \mathbb{R} \times K_6$. Thus these G_2 holonomy results encompass the previous CY results.

The corrected gravitino transformation was "cooked up" to retain supersymmetry in the corrected G_2 background. We must check that it at least admits a covariant Riemannian generalisation, that does not make use of special tensors peculiar to G_2 backgrounds. This is more restrictive than the previous CY case. Remarkably, the previous 6-Gamma Riemmanian expression still works.

Corrections to (Minkowski)$_4 \times K_7$ in M-Theory

So far, we have considered α'^3 corrections at tree-level in type II string theory. The IIA string is an S^1 compactification of M-theory. All the tree-level α'^3 corrections vanish in the limit of uncompactified M-theory. There *are* (Riemann)^4 corrections in M-theory, which correspond to one-loop α'^3 corrections in the IIA string.

At one loop, the α'^3 corrections in the type IIA and type IIB string differ, because of different R-R sectors circulating in the loop. In type IIB, $SL(2,Z)$ duality implies it is the same as at tree-level:

$$Y \propto (t^{i_1\cdots i_8}\, t^{j_1\cdots j_8} - \tfrac{1}{4}\epsilon^{i_1\cdots i_8}\,\epsilon^{j_1\cdots j_8})\, R_{i_1 i_2 j_1 j_2}\cdots R_{i_7 i_8 j_7 j_8} = Y_0 - E_8$$

(with no $e^{-2\phi}$ factor, since it is 1-loop). In type IIA, we have instead

$$\widetilde{Y} \propto (t^{i_1\cdots i_8}\, t^{j_1\cdots j_8} + \tfrac{1}{4}\epsilon^{i_1\cdots i_8}\,\epsilon^{j_1\cdots j_8})\, R_{i_1 i_2 j_1 j_2}\cdots R_{i_7 i_8 j_7 j_8} = Y_0 + E_8$$

and in addition there is a Chern-Simons term $B_2 \wedge t\, R^4$. These lift to terms of the form $\beta(\hat{Y}_0 + \hat{E}_8)$ and $\beta\, \hat{A}_3 \wedge t\, \hat{R}^4$ in M-theory. ($\beta \sim \alpha'^3$.)

Eleven-Dimensional Lagrangian

There are also one-loop α'^3 corrections to the form-field Lagrangian terms, whose detailed structure is unknown. This prevents one from considering corrections to backgrounds in string or M-theory with fluxes where such terms would contribute at this order. But we don't need to know about such terms in order to discuss corrections to M-theory backgrounds that are purely gravitational at leading-order.

For $(\text{Minkowski})_4 \times K_7$ backgrounds, where the curvature is restricted to seven dimensions, neither the $\hat{E}_8$ term nor the $\hat{A}_3 \wedge t\,\hat{R}^4$ term contribute at this order. It would suffice to consider the eleven-dimensional Lagrangian

$$\mathcal{L} = \sqrt{-\hat{g}}\left(\hat{R} - \frac{\beta}{1152}\,\hat{Y}_0\right)$$

However, it is more elegant to exploit the freedom to add terms that vanish by the leading-order field equations. Adding these terms does not change the physics in any way; it corresponds merely to making field redefinitions. But, as is often the case, there exist more convenient, and less convenient, choices of field variable.

M-Theory Equations for (Minkowski)$_4 \times K_7$

Performing the field redefinition

$$\hat{g}_{MN} \longrightarrow (1 + \frac{\beta}{5184}\hat{R}\,\hat{S})\,\hat{g}_{MN}$$

where $\hat{S}$ is the 6-dimensional Euler integrand, leads to

$$\mathcal{L} = \sqrt{-\hat{g}}\left(\hat{R} - \frac{\beta}{1152}(\hat{Y}_0 - \hat{R}\,\hat{S})\right)$$

up to order β. Variation yields the equations of motion

$$\begin{aligned} \hat{R}_{\mu\nu} - \tfrac{1}{2}\hat{R}\hat{g}_{\mu\nu} &= -\frac{\beta}{1152}\Box S \\ \hat{R}_{ij} - \tfrac{1}{2}\hat{R}\hat{g}_{ij} &= \frac{\beta}{1152}(X_{ij} + \nabla_i\nabla_j S - g_{ij}\,\Box S) \end{aligned}$$

Thus $d\hat{s}^2_{11} = \eta_{\mu\nu}\,dx^\mu\,dx^\nu + ds^2_7$, where the metric on K_7 again satisfies

$$R_{ij} = \frac{\beta}{1152}(\nabla_i\nabla_j\,S + c_{ikm}\,c_{j\ell n}\,\nabla^k\nabla^\ell\,Z^{mn})$$

The field redefinition has compensated for the absence of the dilaton, allowing us to get the identical K_7 field equation as in string theory.

Corrections to (Minkowski)$_3 \times K_8$ in M-Theory

With curvature in 8 dimensions, the $\beta\,\hat{E}_8$ and $\beta\,\hat{A}_3 \wedge t\,\hat{R}^4$ terms contribute at order β to corrections to gravitational backgrounds. The relevant M-theory Lagrangian is
$\mathcal{L} = \hat{R}\hat{*}\mathbf{1} - \frac{1}{2}\hat{*}\hat{F}_4 \wedge \hat{F}_4 - \frac{1}{2}\hat{A}_3 \wedge \hat{F}_4 \wedge \hat{F}_4 + \mathcal{L}_1$ with

$$\mathcal{L}_1 = -\frac{\beta}{1152}\,(\hat{Y}_0 + \hat{E}_8 - \hat{R}\,\hat{S})\,\hat{*}\mathbf{1} + \beta\,(2\pi)^4\,\hat{A}_3 \wedge \hat{X}_8$$

(including the redefinition-dependent term for convenience), where

$$\hat{X}_8 = \frac{1}{192(2\pi)^4}\,[\mathrm{tr}\hat{\Theta}^4 - \tfrac{1}{4}(\mathrm{tr}(\hat{\Theta})^2)^2]$$

This implies the field equations

$$\begin{aligned}
\hat{R}_{\mu\nu} - \tfrac{1}{2}\hat{R}\,\hat{g}_{\mu\nu} &= -\frac{\beta}{1152}\,(\Box S + E_8)\,g_{\mu\nu} \\
\hat{R}_{ij} - \tfrac{1}{2}\hat{R}\,\hat{g}_{ij} &= \frac{\beta}{1152}\,(X_{ij} + \nabla_i\nabla_j\,S - g_{ij}\,\Box S) \\
d\hat{*}\hat{F}_4 - \tfrac{1}{2}\hat{F}_4 \wedge \hat{F}_4 &= (2\pi)^4\,\beta\,X_8
\end{aligned}$$

where as usual X_{ij} is the variation of $\sqrt{-g}(Y_0 - E_8)$ in K_8. The Ricci-flatness of K_8 is corrected to $R_{ij} = (\beta/1152)(X_{ij} + \nabla_i\nabla_j S)$, and we take

$$
\begin{aligned}
d\hat{s}_{11}^2 &= e^{2A}\,\eta_{\mu\nu}\,dx^\mu\,dx^\nu + e^{-A}\,ds_8^2 \\
\hat{F}_4 &= d^3x \wedge df + G_4
\end{aligned}
$$

If we assume for now that K_8 is non-compact we can take G_4, which lives in K_8, to be zero, and the field equations then imply

$$
\Box A = \frac{\beta}{1728} E_8\,, \qquad \Box f = \beta\,(2\pi)^4 *X_8
$$

Note that the non-zero warp factor is forced by the $\beta(\hat{Y}_0 + \hat{E}_8 + \cdots)$ correction, while the non-zero $\hat{F}_4$ is forced by the anomaly term $\beta \hat{A}_3 \wedge \hat{X}_8$.

The anomaly term is $X_8 = 1/(192(2\pi)^4)\,(P_1^2 - 4P_2)$, where P_i is i'th Pontrjagin class. It was shown (Isham & Pope 1988) that if an 8-manifold admits a nowhere-vanishing spinor (as does a special-holonomy manifold with its covariantly-constant spinor) then there is a topological relation $P_1^2 - 4P_2 = 8\chi$, and hence we have $E_8 = 576(2\pi)^4 *X_8$. This leads to

$$
f = 3A
$$

This is the same relation one finds in a standard M2-brane solution.

Supersymmetry of Corrected $(\text{Minkowski})_3 \times K_8$

Taking the leading-order K_8 to have Spin(7) holonomy, we find its Ricci-flatness is corrected to

$$R_{ij} = \frac{\beta}{1152}\big(\tfrac{1}{2}c^{mn}{}_{k(i}\, c^{pq}{}_{\ell j)}\, \nabla^k \nabla^\ell\, Z_{mnpq} + \nabla^k \nabla_\ell\, Z_{mnk(i}\, c^{mn\ell}{}_{j)} + \nabla_i \nabla_j\, S\big)$$

where $c_{ijk\ell}$ is the calibrating 4-form of the Spin(7) background and

$$Z^{mnpq} = \tfrac{1}{64}\, \epsilon^{mni_1\cdots i_6}\, \epsilon^{pqj_1\cdots j_6}\, R_{i_1 i_2 j_1 j_2}\, R_{i_3 i_4 j_3 j_4}\, R_{i_5 i_6 j_5 j_6}$$

We can again look for a corrected covariant derivative, whose integrability condition yields this corrected Einstein equation. We find

$$D_i \equiv \nabla_i + \frac{\beta}{1152} Q_i = \nabla_i + \frac{\beta}{4608} c_{ijk\ell}\, \nabla^j\, Z^{k\ell mn}\, \Gamma_{mn}$$

This reduces to our previous G_2 result if $K_8 = R \times K_7$. It also "Riemannianises" to the same 6-Gamma expression as before! So we can find a "corrected covariantly-constant" spinor in the corrected K_8 background.

The gravitino transformation rule is then

$$\delta\hat{\psi}_M = \hat{\nabla}_M\hat{\epsilon} + \frac{\beta}{1152}\hat{Q}_M\,\hat{\epsilon} - \frac{1}{288}\hat{F}_{N_1\cdots N_4}\,\hat{\Gamma}_M{}^{N_1\cdots N_4}\,\hat{\epsilon} + \tfrac{1}{36}\hat{F}_{MN_1\cdots N_3}\,\hat{\Gamma}^{N_1\cdots N_3}\,\hat{\epsilon}$$

with $\hat{Q}_\mu = 0$ and $\hat{Q}_i = Q_i$. Collecting all the contributions up to order β, the "M2-brane relation" $f = 3A$ cancels spin-connection terms against field-strength terms (in a standard M2-brane fashion), and we find a Killing spinor

$$\hat{\epsilon} = e^{\frac{1}{2}A}\,\epsilon \otimes \eta$$

where ϵ is constant in $(\text{Minkowski})_3$ and η satisfies the corrected covariant constancy condition

$$\nabla_i\eta + \frac{\beta}{4608}c_{ijk\ell}\,\nabla^j\,Z^{k\ell mn}\,\Gamma_{mn}\eta = 0$$

in the corrected internal space K_8. (Previous discussions omitted the $\mathcal{O}(\beta)$ correction to the gravitino transformation rule, and some omitted the contribution from the $\sqrt{-g}(Y_0 + E_8)$ correction to the Einstein equations.)

(Minkowski)$_3 \times K_8$ Solutions with Compact K_8

With K_8 non-compact, we could take G_4 in $\hat{F}_4 = d^3x \wedge df + G_4$ to be zero. Adding in a flux G_4 in K_8 is optional, provided that it is taken to be sufficiently small that $\mathcal{O}(\beta)$ corrections involving quadratic and higher powers of $\hat{F}_4$ in M-theory are unimportant (since we don't know in detail what they are). Hawking & Taylor-Robinson, and Becker & Becker, have studied the conditions for a solution, with preserved supersymmetry, having $G_4 \neq 0$. The conclusion is that it can be added provided G_4 is self-dual, and that

$$G_{ijk\ell}\,\Gamma^{jk\ell}\,\eta = 0$$

where η is the Killing spinor on K_8.

If K_8 is compact, with non-zero Euler number, the inclusion of G_4 becomes obligatory. This is seen by integrating $d\hat{*}\hat{F}_4 + \frac{1}{2}\hat{F}_4 \wedge \hat{F}_4 = \beta(2\pi)^4\,\hat{X}_8$ over K_8, yielding

$$\int_{K_8} G_4 \wedge G_4 = \frac{\beta(2\pi)^4}{12}\,\chi$$

Corrections to (Minkowski)$_1 \times K_{10}$ in M-Theory

There is one further special-holonomy case that can be studied, where the curved background is a Ricci-flat Kähler 10-manifold, with $SU(5)$ holonomy. This cannot form part of a vacuum in perturbative string theory, which has only nine spacelike dimensions. It can in principle occur in M-theory. It is an interesting example, because it probes aspects of M-theory that cannot be probed directly from light-cone studies in perturbative string theory. However, there is good reason to think that the $\mathcal{O}(\beta)$ corrections in M-theory are valid in a genuinely eleven-dimensional sense.

The eleven-dimensional Einstein equations, with their $\mathcal{O}(\beta)$ corrections, are given by

$$\begin{aligned}\widehat{R}_{00} - \tfrac{1}{2}\widehat{R}\,\widehat{g}_{00} &= -\frac{\beta}{1152}\Box S\, g_{00} + \frac{\beta}{576}\widehat{E}_{00}\\ \widehat{R}_{ij} - \tfrac{1}{2}\widehat{R}\,\widehat{g}_{ij} &= \frac{\beta}{1152}(X_{ij} + \nabla_i\nabla_j S - g_{ij}\Box S) + \frac{\beta}{576}\widehat{E}_{ij}\end{aligned}$$

after imposing the (Minkowski)$_1 \times K_{10}$ Ricci-flat Kähler background conditions in the correction terms on the right-hand sides. Here $\widehat{E}_{MN}$, coming from the variation of the 8-dimensional Euler integrand $\widehat{Y}_2$, is given by

$$
\begin{aligned}
\hat{E}_{00} &= \tfrac{1}{2} Y_2\,, \\
\hat{E}_i{}^j &= E_i{}^j \equiv -\frac{9!}{2^9}\,\delta^{j j_1 \cdots j_8}_{i i_1 \cdots i_8}\, R^{i_1 i_2}{}_{j_1 j_2} \cdots R^{i_7 i_8}{}_{j_7 j_8}\,, \qquad (1)
\end{aligned}
$$

in the (Minkowski)$_1 \times K_{10}$ background. Expecting a warped deformation, we make the ansatz

$$
d\hat{s}_{11}^2 = -e^{2A}\, dt^2 + e^{-\frac{1}{4}A}\, ds_{10}^2\,, \qquad \hat{F}_4 = G_3 \wedge dt + G_4
$$

taking, initially, only G_3 non-zero in K_{10}. The corrected equations of motion then imply

$$
\begin{aligned}
R_{ij} &= \frac{\beta}{1152}\Big(\nabla_{\hat{i}} \nabla_{\hat{j}}\, S + \nabla_i \nabla_j\, S + 2 E_{ij} + \tfrac{1}{4} E_8\, g_{ij}\Big) \\
\Box A &= \frac{\beta}{1728}\, E_8 \\
d{*}G_3 &= (2\pi)^4\, \beta\, X_8
\end{aligned}
$$

The Ricci tensor becomes, as usual, non-zero, and the equation for the warp factor has the 8-dimensional Euler integrand E_8 as its source. The equation for the 3-form G_3 on K_{10} is integrable, and has X_8 as source.

Supersymmetry and (Minkowski)$_1 \times K_{10}$ Solutions

For deformed special-holonomy solutions (Minkowski)$_2 \times K_8$, we got away with making one universal addition, at order α'^3 (or β), to the gravitino transformation rule. This was first deduced by supposing the continued supersymmetry of deformed Calabi-Yau solutions (Minkowski)$_4 \times K_6$. It is highly non-trivial that a Riemannian gravitino correction is possible, that implies continued supersymmetry of all the deformed special-holonomy solutions. Since complete and explicit results for the transformation rules up to order α'^3 are not available, this is the best we have.

We certainly expect further correction terms, but presumably they would play no rôle for $SU(3)$, G_2 or Spin(7) backgrounds. It would have been quite possible that extra terms were needed for (Minkowski)$_1 \times K_{10}$ backgrounds. Remarkably, however, the existing corrected gravitino transformation rule implies that supersymmetry is again preserved in the (Minkowski)$_1 \times K_{10}$ background.

Generalised Holonomy

Generalised holonomy was introduced (Duff, Liu, Hull,...) as a way to characterise the occurrence of Killing spinors in supergravity backgrounds with fluxes, which modify the usual Riemannian spin connection in the supersymmetry transformation rule

$$\delta\psi_\mu = \nabla_\mu \epsilon \qquad \longrightarrow \qquad \delta\psi_\mu = \mathcal{D}_\mu \epsilon$$

The integrability condition $[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = 0$ extends the usual generators $R_{\mu\nu ab}\Gamma^{ab}$ of Riemannian holonomy to an enlarged set of generators (with more Gamma-matrix structures) of generalised holonomy.

We can also employ the idea of generalised holonomy in the string-corrected backgrounds discussed earlier. In cases where the original special-holonomy manifold has dimension ≤ 7, there will be no flux contributions at all, and the generalised holonomy comes just from the string-corrected supersymmetry transformation rule

$$\mathcal{D}_\mu = \nabla_\mu + \tfrac{3}{4} c\, \alpha'^3\, \nabla_\nu R^\rho{}_{\mu ab}\, R^\nu{}_{\sigma cd}\, R^\sigma{}_{\rho ef}\, \Gamma^{abcdef}$$

The commutators of these 2-Gamma and 6-Gamma matrices generate also 10-Gamma matrices (dual to 1-Gamma), so in M-Theory the algebra closes on $\{\Gamma_a, \Gamma_{ab}, \Gamma_{a_1\cdots a_6}\}$.

Generalised Structure Group in M-Theory

Splitting $a = (0, i)$, we have

$$\begin{array}{rcllll} \text{Hermitean} & : & \Gamma_i\,, & \Gamma_{0i}\,, & \Gamma_{0i_1\cdots i_5}\,, & 10+10+252=272 \\ \text{anti-Hermitean} & : & \Gamma_0\,, & \Gamma_{ij}\,, & \Gamma_{i_1\cdots i_6}\,, & 1+45+210=256 \end{array}$$

This describes the maximally-split algebra $Sp(32)$, with $256+16$ non-compact geerators (16 Cartan) and 256 compact generators. This is the *Generalised Structure Group* for purely gravitational backgrounds in M-theory. (It is $SL(16,\mathbb{R})$ in string theory.) In specific backgrounds, we can then find the *Generalised Holonomy Group* as the subgroup realised by the non-vanishing generators.

Inclusion of the 4-form as well enlarges the generalised structure group to $SL(32,\mathbb{R})$. This is unaltered by the presence of the higher-order curvature correction.

We may also consider the *Generalised Transverse Structure Group* and the *Generalised Transverse Holonomy Group*. Here, we factor out the Minkowski spacetime in a corrected Minkowski $\times\, K$ background, and just focus on the curved transverse space K.

We shall only include the contribution of form-field fluxes if these are *forced* by the α'^3 string corrections or M-theory corrections.

Example: Seven-Dimensional Transverse Space

Consider the example of a 7-dimensional transverse space. The deformed background is purely gravitational, and since $\Gamma_{a_1\cdots a_6}$ is dual to Γ_a, we have closure on $\{\Gamma_a, \Gamma_{ab}\}$. These generate $SO(8)$. This is the *Generalised Transverse Structure Group*.

We can determine the *Generalised Transverse Holonomy Group* for an originally-G_2 transverse space by explicit computation of the α'^3 curvature contribution in $\mathcal{D}_\mu$. We took the example of cohomogeneity-1 metrics with $S^3 \times S^3$ principal orbits. Calculation shows the generalised transverse holonomy is $SO(7)$.

So at leading (uncorrected) order, the Killing spinor is the singlet in

$$SO(7) \longrightarrow G_2 : \quad 8 \longrightarrow 7 + 1$$

After including the α'^3 corrections, the Killing spinor is the singlet in

$$SO(8) \longrightarrow SO(7) : \quad 8 \longrightarrow 7 + 1$$

General Dimensions of Transverse Space

Original special holonomy groups, generalised transverse structure groups and generalised tranverse holonomy groups for backgrounds Minkowksi $\times K_n$ in M-theory:

n	Orig. Hol.	GT Struct.	GT Hol.
6	$SU(3)$	$SO(6) \times U(1)$	$SU(3) \times U(1)$
7	G_2	$SO(8)$	$SO(7)$
8	Spin(7)	$SO(8)_+ \times SO(8)_-$	$SO(8)_+ \times$Spin$(7)_-$
10	$SU(5)$	$SL(16, \mathbb{C})$	H

For K_{10}, an example gives

$$H = [U(1) \times SL(5, \mathbb{C}) \times SL(5, \mathbb{C})] \ltimes [\mathbb{C}_1^{(10,1)} \oplus \mathbb{C}_3^{(10,5)}]$$

(Subscripts are $U(1)$ charges.)

Conclusions

- Special-holonomy backgrounds are important in string and M-theory, since they can describe supersymmetric ground states.
- In all except the special case of hyper-Kähler backgrounds, the special holonomy is either reduced (e.g. $SU(3) \to U(3)$) or completely destroyed (e.g. $G_2 \to SO(7)$) by the $\mathcal{O}(\alpha'^3)$ corrections.
- In a generalised sense, however, the structure of the special holonomy group survives; the α'^3-corrected gravitino transformation rule still implies the existence of Killing spinors. Generalised holonomy.
- It would be interesting to verify that the recent complete derivation of the α'^3-corrected gravitino transformation rule confirms this.
- $(\text{Minkowski})_1 \times K_{10}$ backgrounds with $SU(5)$ holonomy provide a rich arena for probing the structure of M-theory.