Existence of non-Landau solutions for Langmuir waves

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The propagation of linear one dimensional (1D) Langmuir waves is reinvestigated using numerical simulations of a new type with very low noise. The dependence of the result on the initial conditions is shown. New solutions are exhibited, with properties different from Landau's, even in the asymptotic behavior, in particular with regard to the damping rate. These solutions are shown to demand a special preparation of the initial plasma perturbation, but in a way which is quite physical, without any singularity in the electron distribution function, contrary to the classical van Kampen's solutions. Using an original theoretical calculation, a simple analytical form is derived for the perturbed distribution function, which allows interpreting both the Landau and non-Landau solutions observed numerically. The numerical results presented and their interpretations are potentially important in several respects: 1) They outline that Landau solutions, for the 1D electrostatic problem in collisionless plasmas, are only a few among an infinite amount of others; even if the non-Landau solutions are much less probable, their existence provides a different view on the concept of kinetic damping and may suggest interpretations different from usual for the subsequent nonlinear effects; 2) they show that the shape of the initial perturbation $\delta f(v)$, and not only its amplitude, is important for the long time wave properties, both linear and nonlinear; 3) the existence of non-Landau solutions makes clear that the classical energy arguments cannot be fully universal as long as they allow deriving the Landau damping rate independently of the initial conditions; 4) the particle signature of Landau damping, different from the usual guess, should imply a change in our understanding of the role of the resonant particles. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2921791]

I. INTRODUCTION

Landau damping is the paradigm of all the wave-particle resonance phenomena. It acts on most plasma waves and has been investigated first for the one dimensional (1D) electrostatic electron plasma oscillations (Langmuir waves). Its linear theory has been established more than 60 years ago by Landau[1] and confirmed experimentally 20 years later.[2] The review,[3] written at the occasion of the 50th anniversary of Landau's original paper, bears witness to the richness of the literature that followed. For all these years until now, Landau's theory, using the Laplace transform and the theory of integration in the complex plane, has been (and remains) the mathematical model for thousands of kinetic studies in collisionless plasmas. Several authors, from the 1960s, have afterward reanalyzed the Landau collisionless damping in different terms, based on energy conservation, with the aim of giving a more physical model. It has been done first in the frame of the Vlasovian theory[4–7] and developed afterward in the Hamiltonian frame.[8,9] The latter approach allowed the authors to go beyond the continuous Vlasov description and study the effects of the discrete nature of the particles. These demonstrations based on energy helped them to elucidate the pivotal role of the so-called "resonant particles"; for given initial conditions, these particles were shown to increase, on average, their kinetic energy at the expanse of the wave energy. These results, if misinterpreted, may lead us to think that there should always be some strong perturbation of the distribution function close to the resonant velocity and that this effect could be considered as the cause of the damping. We show hereafter that this common idea is erroneous for the ordinary *linear* Landau damping; such a perturbation can only occur as a *nonlinear* consequence of the wave or for "non-Landau" solutions.

Landau's solution, for arbitrary initial conditions, is *a priori* only one solution among an infinite amount. From the very beginning of his original paper,[1] Landau has indeed already mentioned this fact and given clear arguments about the infinite number of degrees of freedom in a collisionless plasma. The Vlasov equation, contrary to fluid theories, should lead to an infinite number of modes and not to one single "Langmuir" mode. Akhiezer *et al.*[10] in their classical textbook, indeed exhibit the possibility of various solutions from particular initial conditions. More recent papers,[8,9] have also shown how much the consideration of all these different modes can be powerful by demonstrating that linear and nonlinear dampings can be successfully treated as interactions between them. The excitation of one single mode among an

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infinite number, or of a "wave packet," obviously depends on the initial conditions. Non-Landau solutions are indeed exhibited in the present paper, their existence being foreseen in a theoretical calculation and demonstrated thanks to numerical simulations. We further investigate which initial conditions can lead to classical Landau damping, and which lead to other asymptotic behaviors. This allows separating the cases where the initial conditions are eventually forgotten through some "phase mixing" from those where they do not.

The theoretical calculation presented in this paper is equivalent to—but different from—Landau's, and directly inspired from van Kampen's theory.[12,13] This calculation, which was not available in the literature hitherto, allows us to provide an explicit and analytical form for the distribution linear perturbation. As any other approach, this one of course shows evidence of the unquestionable role of the resonant velocity in the physics of the plasma wave;[4,7,8,14] but it further allows explaining why this role generally does not consist of building any apparent special feature close to the resonant velocity. This calculation also has the great advantage of pointing out the predominant role, for the deformation of the distribution function, of the "ballistic" component (see Sec. III) of the particle trajectories.

Concerning the numerical simulation, looking to the shape of the distribution functions that underlie the propagation and damping of the modes also allows us to investigate the role of the resonant particles. When some localized perturbation in velocity space can actually be shown at the resonant velocity, we can test whether this effect is a nonlinear consequence of the wave or the very cause of its linear damping. We can do that by simply using a linearized version of the code and look at the changes in the damping rate.

The linear theory has motivated very few recent studies. In one of the most recent, Li and Spies[15] have analyzed in detail the particular case of wavelengths smaller than the Debye length. This case is indeed already included in Landau treatment, but deserves special attention since it is quite far from the situation considered in most energy-based demonstrations. In these demonstrations, the phase velocity is always supposed to be much larger than the thermal velocity, which allows treating the resonant particles and the plasma core differently.

On the contrary, nonlinear studies are still numerous. It is indeed worth noticing that, even for small amplitude initial perturbations, the linear phase is confined to times such as $\omega_b t \ll 1$, where ω_b is the trapping frequency (proportional to the square root of the wave amplitude). Afterward, the behavior is always dominated by nonlinear effects. O'Neil[16] has first shown evidence of the possibility that this long time solution tends toward an undamped oscillation of the BGK type; the subject is obviously not decoupled from ours, even if it implies trapped particles. The ballistic effects were already noticed to be of pivotal importance in this study since the route toward such states from given initial conditions were interpreted as due to the phase space folding (or filamentation). It was argued that it gives rise to some ergodization, leading eventually to a "coarse grained" distribution becoming eventually constant with time. Many papers have afterward developed the study of the nonlinear Landau damping in the same frame; the stationary solutions themselves (e.g., Ref. 17) and the ways to reach them from initial conditions (e.g., Ref. 18). Some papers also looked for other possible asymptotic (nonstationary) regimes.[19,20] In this nonlinear context, the initial amplitude has been shown to play a crucial role for the long time behavior (e.g., Refs. 21 and 22), the vicinity of the linear case being specifically analyzed in Ref. 7 as a function of the small parameter ω_b/γ. Nevertheless, little attention has been paid hitherto to its dependence on the initial shape $\delta f(v,x)$, which is investigated here. More recently, several papers such as Ref. 23 questioned the ergodization process invoked by Ref. 16. Because the folding effect eventually leads to unphysical solutions with infinitely thin filaments, the most recent papers go beyond Vlasov theory, and face the complete N-particle problem. They show, with Hamiltonian techniques, that the chaos at the boundaries in phase space plays a crucial role in the long time behavior.[8,9,14,24] The first two papers also go back over the linear side of the question. A similar non-Vlasovian approach furthermore recently allowed us to predict the different long time behaviors by a relatively simple Fermi model.[25]

We shall restrict ourselves here intentionally to linear phenomena; it allows us to investigate the effects that are present for all waves of finite, as well as infinitely small amplitudes, and to clearly separate the very *causes* of the damping from the various nonlinear *consequences* of the waves, which are amplitude dependent. The nonlinear extension of the results presented here should be the subject of further works.

Many numerical simulations have already allowed for the investigation of finite amplitude Langmuir waves, their propagation and damping, their long time behavior and the kinetic nonlinear effects, such as trapping and quasilinear effects (see, e.g., Ref. 11, and references therein). Nevertheless, investigating the linear stage itself remains until now a difficult task; with the same number of particles, ordinary particle in cell (PIC) codes are too noisy for the simulation of very low amplitude waves and it is not possible at all to make any diagnostics of the perturbation of the distribution function. Vlasov codes allow for diagnostics of the perturbation of the distribution function because their noise level is arbitrarily low, but they demand a very large number of meshes to cover all the phase space with a sufficient resolution, notably in the velocity space. This point is especially difficult when the resonant velocity lies in the far tail of the distribution (those corresponding to weak damping) and when the ballistic effects build finer and finer structures in velocity. These reasons partly explain why no analysis of the particle signatures has ever been presented hitherto for linear Landau damping. The perturbative PIC code used in this paper, with a low noise level, and no discretization in the velocity space, allows for a detailed inspection of the phase space all along the linear damping process. But this code also enables us to control these distributions precisely when we set the initial conditions of the simulations. This allows us to set what kind of Langmuir wave we want (controlling the direction of its propagation) and to set "nonstandard" initial conditions. (This last point is especially important in Sec. V.)

The perturbative code used is briefly presented in Sec. II. We show in Sec. III, that "standard" initial conditions in the simulation lead to Landau damping, with a very good accuracy. We show that the form of $f_1(x,v,t)$ does not support the classical view that linear Landau damping is carried by a remarkable feature in the perturbed distribution function around the resonant velocity. In Sec. IV, we rederive the theory of the "kinetic Langmuir mode" and its damping. This derivation does not require the use of complex velocities; it allows revealing more clearly the concepts of ballistic effects and explains the absence of visible "resonant particles behavior" in the standard Landau simulations. In Sec. V, we show that Langmuir waves with damping different from Landau's can be predicted in theory and that they can indeed be observed in the simulations. These examples must be specially prepared, but they do correspond to quite regular distributions and not to pathological ones. We discuss and conclude in Sec. VI.

II. THE PERTURBATIVE PIC CODE

We have used a PIC electrostatic code where the ions are treated as a neutralizing background. With ordinary PIC codes, as the number of particles in the simulation (called macroparticles) is much lower than the number of electrons in a real plasma, the noise level is generally higher in the simulation that in reality by several orders of magnitude. This is not favorable for a detailed study of the linear Landau damping, especially when we are interested in the analysis of the distribution function perturbation f_1 that is ideally considered as of infinitesimal amplitude. Fortunately, such an analysis can be performed through the use of a perturbative PIC code,[28] where the macroparticles are used to represent only the perturbation δf of the particle distribution, and not the whole particle distribution $f=f_0+\delta f$.[26,27] (Here we use the notation $f=f_0+\delta f$ in general, and $f=f_0+f_1$ when f results from linearized equations.) Hence, the noise caused by the low number of macroparticles only affects the perturbation. If $\delta f \ll f$, as is the case for the linear Langmuir wave, the noise in perturbative PIC simulations is reduced by orders of magnitude. Detailed analysis of the Landau damping and diagnostics of $\delta f(x,v)$ are thus possible.

Let us split the distribution function into an equilibrium contribution $f_0(v)$ and a perturbation $\delta f(x,v,t)$. The evolution of δf is given by

$$\frac{\partial \delta f}{\partial t} + \mathbf{v}\cdot\frac{\partial \delta f}{\partial \mathbf{x}} + \frac{d\mathbf{v}}{dt}\cdot\frac{\partial \delta f}{\partial \mathbf{v}} = -\,\delta\mathbf{a}\cdot\frac{\partial f_0}{\partial \mathbf{v}}, \tag{1}$$

where (in the full particle dynamic case) the acceleration term is given by[28] $d\mathbf{v}/dt = d\mathbf{v}_0/dt + \delta\mathbf{a}$ and

$$d\mathbf{v}_0/dt = \frac{q}{m}\mathbf{E_0} = \mathbf{0}, \tag{2}$$

$$\delta\mathbf{a} = \frac{q}{m}\delta\mathbf{E} = \frac{q}{m}\mathbf{E}. \tag{3}$$

This set of equations is *not* linearized. The link between δf and the macroparticles is given by the macroparticle distribution function

$$g(\mathbf{x},\mathbf{v}) = \sum_i S_p(\mathbf{x}-\mathbf{x_i},\mathbf{v}-\mathbf{v_i}) \tag{4}$$

that is related to a shape function S_p, that operates in phase space. We introduce for each particle i a statistical weight w_i,

$$w_i(t) = \frac{\delta f[\mathbf{x_i}(t),\mathbf{v_i}(t),t]}{g[\mathbf{x_i}(t),\mathbf{v_i}(t)]}, \tag{5}$$

where $g[\mathbf{x}_i(t),\mathbf{v}_i(t)] = \Sigma_j S_p[\mathbf{x}_j(t)-\mathbf{x}_i(t),\mathbf{v}_j(t)-\mathbf{v}_i(t)]$. From Eq. (1), it is possible to derive the evolution of the shape function at each time step through the relation[28]

$$\frac{dw_i}{dt} = -\,\delta\mathbf{a}\frac{1}{g}\frac{\partial f_0}{\partial \mathbf{v}} = -\,\frac{q}{m}\mathbf{E}\left(\frac{f}{g}-w_i\right)\frac{1}{f_0}\frac{\partial f_0}{\partial \mathbf{v}}. \tag{6}$$

Usual (nonperturbative) PIC simulations can be recovered from a perturbative code, just by setting $w_i=1$. The choice for the initialization of g (and consequently of f/g and w_i) is open. We use, in this paper, a function g with a uniform probability for particle velocities comprised between two limits v_m and v_M, and null outside ("waterbag distribution"). Then we set f/g with the same profile as f, provided that $|v_m|$ and v_M are large enough. Such initial conditions allow for a very good description of the dynamics of the tail of the distribution function in spite of a reduced number of macroparticles.

Such codes are nonlinear, but it is also possible to make a linear version of them, which solves the linearized Vlasov equation, i.e., make $\delta f = f_1$ (the linear perturbations are marked with an index "1"):

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{x}} + \frac{d\mathbf{v}_0}{dt}\cdot\frac{\partial f_1}{\partial \mathbf{v}} = -\,\frac{q}{m}\mathbf{E}\cdot\frac{\partial f_0}{\partial \mathbf{v}}. \tag{7}$$

In the linear case, f_0/g is invariant along a (ballistic) macroparticle trajectory.

III. SIMULATION OF LINEAR LANDAU DAMPING

As we shall see in the following results of numerical simulations, the "ballistic effects" are quite basic and by far the most evident when looking to the phase space. Let us therefore first define what we call "ballistic effect" before going to the results themselves. It is illustrated in Fig. 1; in a collisionless plasma, if the phase space is initially perturbed in x as shown in the lower plot, and if each particle simply keeps its initial velocity, the difference in velocities between the upper part and the lower part of the figure makes the perturbation fold more and more with time. For a given abscissa x, this results in a perturbation $\delta f(v)$ that is modulated with a shorter and shorter period in v. This pure ballistic effect is of course modified when the trajectories undergo the influence of the self-consistent electric field, which makes the very existence of the collisionless wave. Nevertheless, we will see that the corresponding modification is not so important and that the basic ballistic effects remain the most visible feature.

In the simulations, reduced variables are used. The unit of velocity is c, the unit of time is the inverse plasma frequency ω_{pe}^{-1}, the unit of electric charge is the electron charge

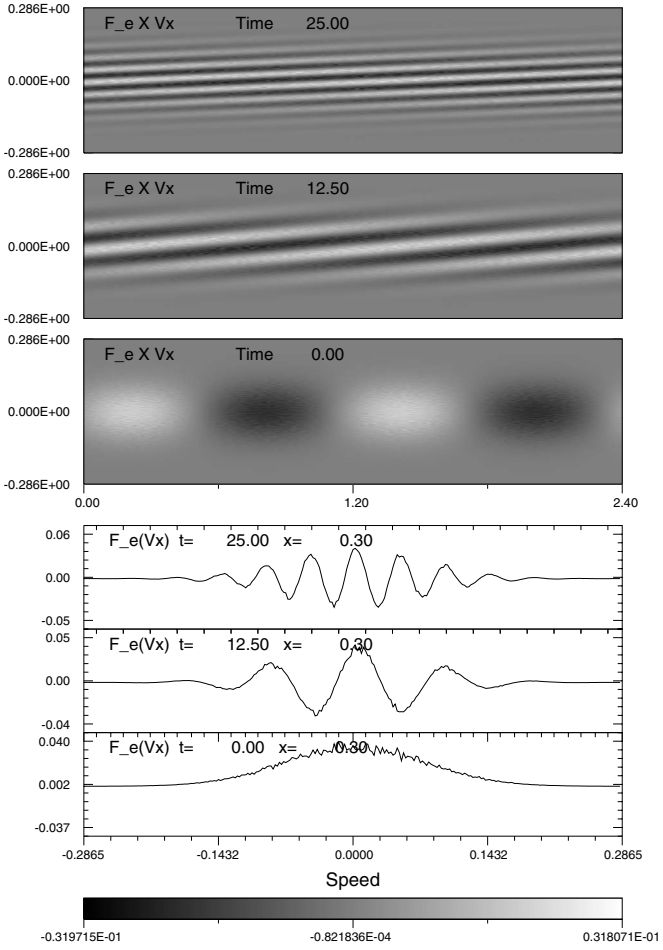

FIG. 1. Upper panel: Phase space density perturbation $\delta f(x,v)$ at three different times for a particle-test simulation where the particles have ballistic motion and the unperturbed distribution function $f_0(v)$ is a uniform Maxwellian. The initial condition corresponds to a sinusoidal perturbation of the density, $\delta f(t=0,x,v)=\alpha f_0(v)\sin(kx+\Phi)$. Lower panel: Cuts $\delta f(v)$ of the above phase space distribution functions, at the same times and $x=0.3$. (The reduced units are defined in Sec. III.)

absolute value e, the unit of mass is the electron mass m_e. In this system, the unit of length is $c\omega_{pe}^{-1}$ and the unit of electric field is $cm_e\omega_{pe}/e$.

The simulations are initialized with sinusoidal perturbations such that the electrostatic field and the plasma velocity and density correspond to the polarization of a linear Langmuir wave in the cold plasma approximation with a positive phase velocity,

$$E_1(x,t=0) = -\hat{E}_1 \cos(kx+\phi), \tag{8}$$

$$n_1(x,t=0) = N_{1,0}(x) = -\frac{k\hat{E}_1}{\omega_L^2}\sin(kx+\phi), \tag{9}$$

$$v_1(x,t=0) = V_{1,0}(x) = -\frac{\hat{E}_1}{\omega_L}\sin(kx+\phi), \tag{10}$$

where ω_L is the Langmuir mode pulsation, estimated theoretically (the same reduced units as in the simulation are used).

The initial distribution function corresponds to these moments and is chosen to be

$$f(v,x,t=0) = f_0 + \delta f \tag{11}$$

with

$$f_0(v,x,t=0) = \frac{n_0}{\sqrt{2\pi}v_t}\exp\left(-\frac{v^2}{2v_t^2}\right), \tag{12}$$

and

$$\delta f(v,x,t=0) = \frac{N_{1,0}(x)}{\sqrt{2\pi}v_t}\exp\left\{-\frac{[v-V_{1,0}(x)]^2}{2v_t^2}\right\}. \tag{13}$$

This initialization is the simplest one. The density and velocity perturbations exactly correspond to the chosen functions $N_{1,0}(x)$ and $V_{1,0}(x)$. However, it does not strictly correspond to a solution of the system since: 1) The finite temperature makes a departure from the cold plasma one and 2) the given initial δf does not have a strictly sinusoidal dependence in x as it should (but E_1, n_1, and v_1 have). A cleaner initialization is quite possible from the theoretical results, but it has not been done hitherto. This one is anyway close enough to obtain what we need; the foregoing evolution mainly consists of one single wave propagating in the positive direction, the backward wave being of much smaller amplitude. Practically, we set the distribution function in the perturbative code thanks to the choice of adequate values of w_i for each particle, and impose $E_1(x,t=0)$.

A Landau damping "reference" simulation (LDR) is described here. We set two wavelengths along the x axis, $k\lambda_D=0.39$, where λ_D is the Debye length. The simulation uses 2000 macroparticles per cell, the grid has 32×4 cells, and $\Delta x=0.075c/\omega_{pe}$. The distribution $f_0(v)$ is Gaussian and the electron thermal velocity is $v_t=0.075c$. The time step is $\omega_p\Delta t=0.05$.

The value ω_L used in the initialization [Eqs. (9) and (10)], is the solution of the classical kinetic dispersion relation (which will be rederived hereafter). This equation provides $\omega_L=1.2756\omega_{pe}$ and $\gamma_L=-0.0609\omega_{pe}$. It can be noted that the approximate value coming from Bohm and Gross equation ($\omega \gg kv_t$) and a small damping rate expansion ($\gamma \ll \omega$) is rather different and insufficiently accurate ($\omega_L'=1.209\omega_{pe}$ and $\gamma_L'=-0.090\omega_{pe}$).

Figure 2 displays the evolution of the electric energy density integrated over the simulation domain. The straight line corresponds to Landau's theoretical solution. We can see that the simulation fits the theoretical curve over more than four decades. The slight oscillations show that the Langmuir wave in the opposite direction is also excited with a small amplitude in the initialization as explained above. The modulation depth observed here corresponds to a ratio about 2 on energy, i.e., a ratio of 15%–20% between the backward/forward wave amplitudes. At later times, the damped wave electric energy stops decreasing. This saturation can be related to the "thermal noise" (or spontaneous emission) due to the finite number of particles, as shown in Ref. 8. It may also be due to parasitic shorter wavelength modes (not damped) injected in the initialization.

Is the wave amplitude sufficiently low to study *linear*

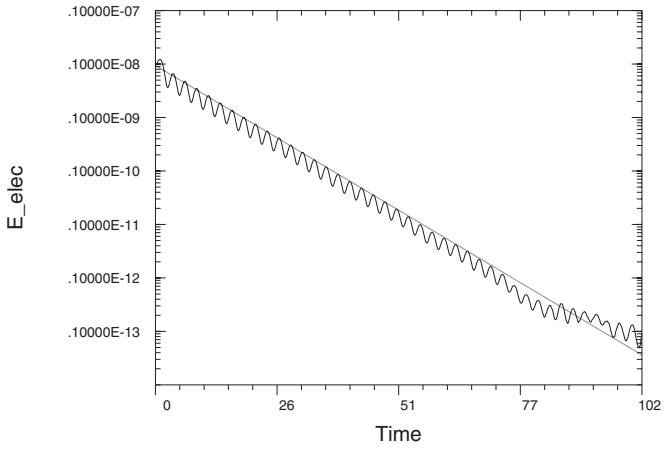

FIG. 2. Time evolution of the electric energy (square of the electric field averaged upon the simulation domain) as a function of time, for the Landau damping reference (LDR). The straight line (of slope $2\gamma_L$) indicates the evolution predicted by the Landau theory.

Landau damping? Answering this question demands us to compare the linear damping rate γ_L to the trapping frequency ω_b. We have set here $\hat{E}_1=10^{-4}$, the amplitude of the corresponding density perturbation is $n_1=3.2\times10^{-4}n_0$. The ratio of the nonlinear trapping frequency over the linear damping rate is therefore $\omega_b/\gamma=\sqrt{k\hat{E}_1}/\gamma=0.37$, which is indeed smaller than unity. When comparing with theoretical results, we cannot nevertheless exclude the influence of small nonlinear effects. Therefore, the present simulation is conducted again, but with the linearized version of the perturbative code. It gives the same results as shown in Fig. 2.

These satisfactory results confirm that the standard Landau damping is very well accounted for in the present perturbative PIC simulations. We are therefore allowed to examine the electron distribution function and look for the particles which are responsible for it. What do we observe at the resonant velocity?

Figure 3 shows the initial distribution function around a given cell (of arbitrary position $x_0=4$) of the simulation grid. The lower panel shows the total distribution function $f_0(v)+\delta f(v,x_0,t=0)$, and the upper panel shows the perturbation δf only. The vertical lines represent the resonant velocities $v_\varphi=\pm\omega_L/k$. We can notice the very low amplitude of the perturbation, and the small value of f_0 at the resonant velocity.

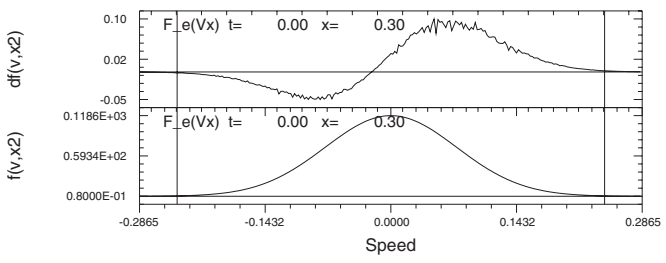

FIG. 3. Initial electron distribution function for the LDR simulation. Lower panel: Total distribution function $f_0(v)+f_1(v,x_0,t=0)$. Upper panel: Perturbation of the distribution function $f_1(v,x_0,t=0)$. The vertical lines represent the resonant velocities $v_\varphi=\pm\omega_L/k$.

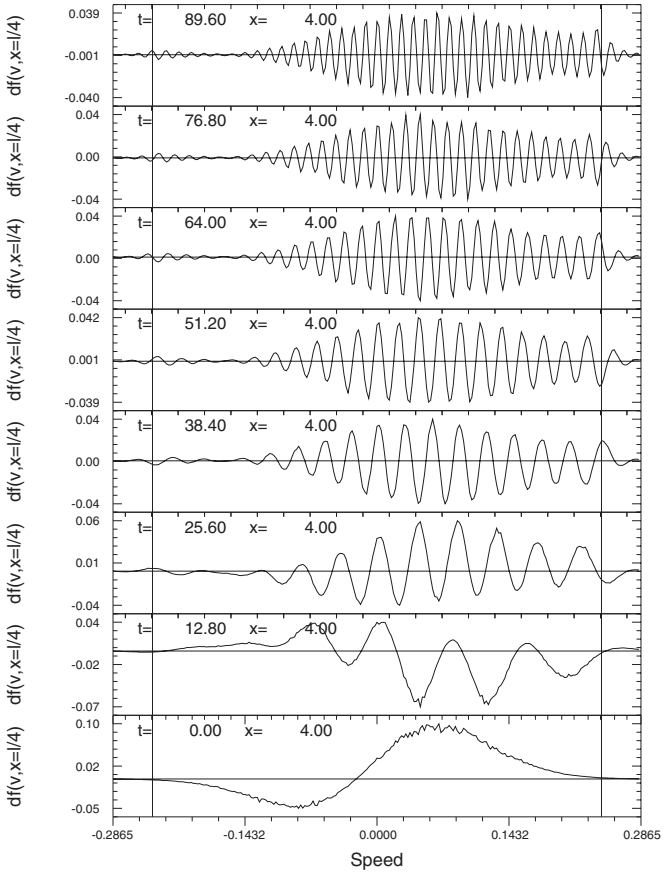

FIG. 4. Perturbation $\delta f(v,x_0,t)$ at various times for the LDR simulation.

The perturbation of the electron distribution function $\delta f(v,x_0)$ around the same cell, at various times is displayed in Fig. 4. Two striking results appear, which contradict usual intuition about Landau damping:

1. Nothing special is observed at, or in the vicinity of, the resonant velocity. Therefore, the concept of resonant particles does not appear to correspond to a special class of particles which would have a noticeable behavior in the distribution function.
2. The overall distribution shows oscillations of the function $\delta f(v)$, with a frequency (in v) increasing with time. This is just what we expect for the ballistic effect explained in the Introduction.

Figure 5 shows the phase spaces at $t=0$ and $t=12.8$ and $t=25.6$ for the Landau damping reference simulation (LDR). The comparison of Fig. 1 and Fig. 5 shows that the effect of the electric field only slightly perturbs the ballistic behavior (second order).

IV. THEORETICAL CALCULATION OF $f_1(x,v,t)$

In order to interpret the results observed in the simulations and to understand the physics underlying them, it is necessary to determine theoretically the distribution function perturbation. This can of course be done from Landau's theory but, as it demands complementary calculations, it is simpler to briefly rederive the complete theory. We present it here in a way which intends to show better evidence of the

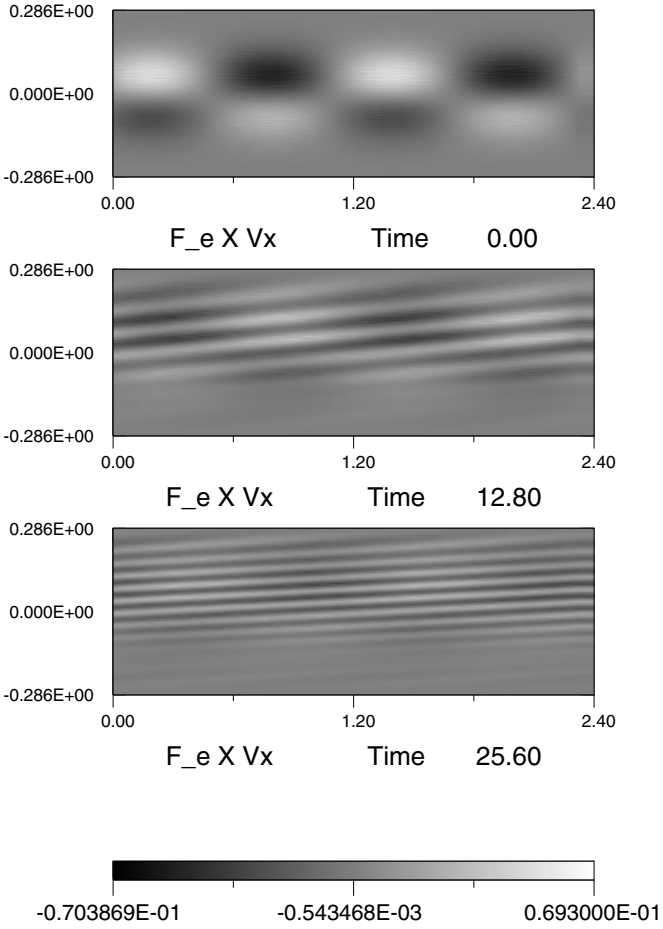

FIG. 5. Electron phase space at various times at the beginning of the Landau damping reference simulation (LDR).

role of the distribution function and avoid some of the mathematical knacks too far from intuition, such as, complex contour deformations in a complexified velocity space.

Our approach is inspired by van Kampen's,[12] which has been shown to be equivalent to Landau's by Case.[13] We will proceed through the following steps:

- Search for real eigenmodes (ω real, no damping). Find an infinite number of them, with singular distributions.
- Build complex eigenmodes (ω complex, damping) by linear superposition of the preceding solutions. Find again an infinite number of solutions, but with regular distributions.
- Choose, within the preceding set, the solution that corresponds to the most likely initial conditions.

Real eigenmodes: We consider electrostatic longitudinal and linear perturbations where the ions are considered as a neutralizing background. We look for solutions with monochromatic variations, that are valid on the whole time axis,

$$\phi_1(x,t) = \hat{\phi}_1 e^{ikx} e^{-ikv_\varphi t}. \tag{14}$$

The spatial variation is fixed by k and the temporal variation is searched via $v_\varphi = \omega/k$. The distribution function perturbation $f_1(v)$ is supposed to have the same monochromatic variation as the electrostatic potential ϕ_1. The electron dynamics is then given by the linearized Vlasov equation,

$$(v - v_\varphi) f_1 = \frac{q\phi_1}{m} f_0'. \tag{15}$$

Notation f_0' stands for $\partial f_0/\partial v$, and $q=-e$ is the negative electron charge.

As outlined by van Kampen,[12] any solution of this linearized Vlasov equation is the sum of two terms, $f_1 = f_{1p} + f_{1b}$. The first term is a particular solution of the complete equation, including its right-hand side. In this part, the ballistic motion is modified by the electric field forcing. There is an infinity of such particular solutions. The most handy one is the distribution f_{1p} defined as

$$f_{1p} = \frac{q\phi_1}{m} f_0' P\left\{\frac{1}{v - v_\varphi}\right\}. \tag{16}$$

As we consider real values of v_φ, the plain function $1/(v-v_\varphi)$ would have a pole in $v=v_\varphi$, which would make its integrals ill-defined when calculating the density or any macroscopic moment. As a particular solution, we therefore choose instead the distribution $P\{1/v-v_\varphi\}$, which specifies that its integrals are to be taken in the sense of the principal parts (see, for instance, Ref. 29). One is free of course to make any other choice, but this one is the simplest.

The second term is the general solution of the homogeneous equation $(v-v_\varphi)f_{1b}=0$. This part of the solution can be told fully "ballistic" since it is obtained by canceling the electric field forcing (kernel of the application $f_1 \to \phi_1$). Its expression is

$$f_{1b} = n_{1b}\delta(v - v_\varphi), \tag{17}$$

where n_{1b} has the same monochromatic dependence as ϕ_1 and f_1. This confirms the important role of the unperturbed (ballistic) motion of particles in the transport of the initial perturbation. With the present hypothesis of monochromatic variations with a real v_φ, one must also note that this fully ballistic part is a Dirac distribution at the resonant velocity $v=v_\varphi$; it means that the only particles able to adapt the solution to the initial condition are then the exactly resonant particles.

For any value of the phase velocity v_φ, the full solution of the linearized Vlasov equation (15) is therefore a well defined distribution of v,

$$f_1 = \frac{q\phi_1}{m} P\left\{\frac{f_0'(v)}{v - v_\varphi}\right\} + n_{1b}\delta(v - v_\varphi). \tag{18}$$

Let us emphasize once again that this decomposition is not unique. The second term ("fully ballistic") complements the particular solution chosen for the first one (here the principal value distribution) and allows us to adapt it to any initial condition on f_1. Its amplitude n_{1b} therefore depends on the choice of this particular solution.

Equation (18) gives the response of the plasma to the electric potential ϕ_1. The system has to then be closed by a second equation giving the dependence of the potential on the plasma. Concerning the present study of Langmuir waves, this equation is simply Poisson's equation, which makes the integro-differential character of the problem,

$$\phi_1 = \frac{q}{k^2 \epsilon_0} n_1 \tag{19}$$

with

$$n_1 = \int_{-\infty}^{+\infty} f_1(v) dv. \tag{20}$$

For studying Landau damping for waves different from Langmuir waves, only this second equation has to be changed, which makes the present calculation general, or at least easy to generalize. For an ion acoustic wave, for instance, the same calculation holds except that the Vlasov equation to be considered is the ion one and that the second equation comes from an isothermal hypothesis on the electrons together with a quasineutrality hypothesis, which provides $\phi_1 = n_1 T_e / q$ instead of Eq. (19).

Integrating Eq. (18) and injecting Eq. (19) allows elimination of the integration constant n_{1b} as a function of the macroscopic perturbation ϕ_1,

$$n_{1b} = n_0 \frac{q\phi_1}{m} D_p(v_\varphi) \tag{21}$$

with

$$D_p(v_\varphi) = \frac{k^2}{\omega_p^2} - P \int_{-\infty}^{+\infty} \frac{f_0'(u)/n_0}{u - v_\varphi} du, \tag{22}$$

and finally obtaining, for any v_φ,

$$f_1 = n_0 \frac{q\phi_1}{m} \left[P\left\{ \frac{f_0'(v)/n_0}{v - v_\varphi} \right\} + D_p(v_\varphi) \delta(v - v_\varphi) \right]. \tag{23}$$

This result calls for several remarks:

1. At the end of this calculation, the phase velocity v_φ remains undetermined. It means that, as expected, we get an infinite number of eigenmodes, with phase velocities that can be arbitrarily prescribed, each of them being excited by a different initial condition on f_1.
2. The perturbed distribution function f_1 is composed of two terms: The first one is a particular solution, forced by the electric field, and is chosen as a principal value distribution, and the second one is the complementary ballistic part. The second term contains the factor $D_p(v_\varphi)$. This clearly indicates that the approximate "fluid" dispersion equation $D_p(v_\varphi) = 0$ (obtained with the only principal part of the integral) amounts to neglecting the adaptable ballistic part. On the contrary, any theory leading to kinetic damping actually takes this part into account in some way.
3. The perturbed distribution function f_1 is singular at the resonant velocity, due to the two terms. This singularity has an odd part coming from the first term (through a pole and a principal part) and an even part from the second (through a Dirac distribution). The solution is therefore unphysical in the sense that it is not excitable by regular initial conditions.

Complex eigenmodes: The infinite number of kinetic eigenmodes that we have just found, and which are sometimes referred to as "van Kampen modes," provide actually a basis on which any solution of the problem having a defined Fourier transform can be decomposed. From given initial conditions, the time evolution is given by a linear superposition of these modes, which is nothing but an inverse Fourier transform on the real ω axis. The reason why it is not a convenient basis is the unphysical form of their f_1: decomposing regular initial conditions on a singular basis is obviously not the easiest way to proceed.

In order to restrict the study to regular solutions, we will now superimpose the previous monochromatic solutions and consider a new basis composed of wave packets. We so extend the notion of "eigenmodes" to solutions where the macroscopic quantities (n_1, ϕ_1) still have a fixed mathematical form, but where this form is no more a plain sinusoid for t from $-\infty$ to $+\infty$; it implies an exponential decrease for $t > 0$. Furthermore, we will no more impose that the distribution $f_1(v)$ characteristic of the microscopic state has the same temporal variation; the form of this temporal variation will be a result of the calculation, as a consequence of the same linear superposition. The form of the macroscopic quantities at $t < 0$ has also to be fixed in order to obtain uniqueness of the solution, but this choice is arbitrary. As a first step, we will simply consider that n_1 and ϕ_1 are null for negative times. This simple choice is strongly reminiscent of the Laplace transform used in Landau's theory, but it is worth noticing that the approach here is rather different from Landau's; we are looking directly to new "eigenmodes" while Landau's treatment solves an initial condition problem. The "eigenmodes" appear in Landau's treatment only at the end of the study, starting from a contour of integration in the upper half of the complex ω-plane for the inverse Laplace transform, deforming it toward the lower half plane, and revealing finally poles of the the integrand in this half plane. Let us outline that the "basis" that we are building here is not a complete basis for all solutions, but only a basis for a subset of them; it excludes the unstable ones, e.g., those which increase infinitely with time. These unstable solutions will not be studied in the present paper but it is likely that they can be approached by a similar method (but with different wave packets).

Let us use here a wave packet defined by

$$\phi_1(x,t) = \Phi_1(x) e^{-ik\tilde{v}_\varphi t} \quad \text{if } t \geqslant 0, \tag{24}$$

$$\phi_1(x,t) = 0 \quad \text{if } t < 0, \tag{25}$$

with $\Phi_1(x) = \hat{\phi}_1 e^{ikx}$ and where $\tilde{v}_\varphi$ is now a complex phase velocity, allowing for an exponential decrease at $t > 0$ defined by $\tilde{v}_\varphi = \tilde{\omega}/k = (\omega + i\gamma)/k$, with $\gamma < 0$.

Knowing the Laplace transform, one easily finds how this temporal form of ϕ_1 can be represented as a linear superposition of the preceding monochromatic (undamped) waves,

$$\phi_1(x,t) = -\frac{\Phi_1(x)}{2i\pi} \int_{-\infty}^{+\infty} e^{-ikv_\varphi t} \frac{dv_\varphi}{v_\varphi - \tilde{v}_\varphi}. \tag{26}$$

One can say that the exponential decrease of the wave packet is indeed due to phase mixing between its monochro-

matic components. Applying the same linear superposition to f_1 given by Eq. (23) leads to the following solution for the distribution function perturbation:

$$f_1 = n_0 \frac{q\Phi_1}{m} \frac{1}{2i\pi(v-\tilde{v}_\varphi)} \times \left[2i\pi \frac{f_0'(v)}{n_0} \left(e^{-ik\tilde{v}_\varphi t} - \frac{1}{2} e^{-ikvt} \right) - D_p(v) e^{-ikvt} \right]. \quad (27)$$

Details of this calculation are given in the Appendix. One important feature is shown in this result: When the macroscopic parameters such as ϕ_1 have a pure exponential evolution as $e^{-ik\tilde{v}_\varphi t}$, the perturbed distribution function f_1 does not have the same behavior. For each velocity, f_1 is the sum of terms varying as $e^{-ik\tilde{v}_\varphi t}$ and terms varying as e^{-ikvt}. The latter correspond to the folding in phase space described in the Introduction and already observed in the simulations, and labeled as "ballistic effects."

If one wants to make this result explicit for the particular example of a Maxwellian distribution function $f_0(v)$ of thermal velocity $v_t=(kT/m_e)^{1/2}$, it can be written as

$$D_p(v_\varphi) = \frac{k^2}{\omega_p^2} - \frac{Z_p'(\xi_\varphi)}{2v_t^2}, \quad (28)$$

where ξ_φ stands for $v_\varphi/\sqrt{2}v_t$ and where the notation Z_p' represents the derivative of a function which is not the classical Fried and Conte function $Z(\xi)$ (which already implies the classical contour deformations), but simply the corresponding principal part, $Z(\xi)=Z_p(\xi)+i\sqrt{\pi}e^{-\xi^2}$. This function is defined without ambiguity and is real for a real variable,

$$Z_p'(\xi_\varphi) = \frac{1}{\sqrt{\pi}} P \int_{-\infty}^{+\infty} \frac{-2\xi e^{-\xi^2}}{\xi - \xi_\varphi} d\xi. \quad (29)$$

Equation (27) is the equivalent for complex modes to Eq. (23) for real ones. It also calls for a few comments:

1. At the end of this calculation, the complex phase velocity $\tilde{v}_\varphi$ remains as undetermined as was the real v_φ in the previous one. It means that we are once again left with an infinite number of solutions, which form a new basis.
2. These new solutions are again composed of two terms. The complementary ballistic term [proportional to $D_p(v)$] does not concern the only strictly resonant particles, since it is now multiplied by a function $e^{-ikvt}/(v-\tilde{v}_\varphi)$ instead of a Dirac peak. It shows that the initial conditions are now transported via the ballistic behavior of a wider range of particles. It is worth noticing also that one of the terms in e^{-ikvt} comes from the first term (i.e., from the particular solution). This shows that the folding showing evidence of the ballistic effects is partly due to this "principal value" part.
3. The perturbed distribution function is no more singular; the denominator $(v-\tilde{v}_\varphi)$ never cancels for real velocities, as soon as $\tilde{v}_\varphi$ is not real, i.e., as γ is not zero. This weakens the common arguments against van Kampen's approach reproaching it for involving singular, and therefore unphysical, distribution functions (e.g., Ref. 3).
4. The choice of a basis always remains partly arbitrary and the above regular solutions can still be combined together to build new ones, which may appear more convenient or more "physical." The result obtained here concerns the choice that we have made of the macroscopic wave packet null at $t<0$. It can easily be generalized. Adding a non-null negative part for the wave packets amounts, in Fourier space, to add poles with positive imaginary parts, i.e., replacing Eq. (26) by

$$\phi_1(x,t) = -\frac{\Phi_1(x)}{2i\pi} \int_{-\infty}^{+\infty} e^{-ikv_\varphi t} \frac{R(v_\varphi)}{R(\tilde{v}_\varphi)} \frac{dv_\varphi}{v_\varphi - \tilde{v}_\varphi}, \quad (30)$$

where R is an analytical function (e.g., a rational fraction) having its poles in the upper half complex v-plane, such as its limits at infinity are finite and $R(\tilde{v}_\varphi)\neq 0$. The same correction of course applies to f_1. This kind of change can be useful for adding new physical constraints to the searched eigenmodes.

Choice of one "kinetic mode": We have an infinite set of regular solutions, which can therefore be considered as an infinite set of "kinetic modes" on which any initial solution can again be decomposed. This new infinity is even double now since we can choose arbitrarily, not only the real part of the phase velocity, but also its imaginary part. What can be called *the* kinetic Langmuir mode among this infinity, and what is its damping rate? In Landau's approach, subject to some conditions which are not fully explicit, one unique dispersion equation is indeed derived. As this unique equation has several solutions, different behaviors are possible after different initial conditions, but one unique "asymptotic behavior" is found, which corresponds to the less damped solution. Let us try to better understand what is the condition leading to this unique dispersion relation.

For each of the complex solutions corresponding to the perturbation f_1 given by Eq. (27), i.e., for one given $\tilde{v}_\varphi$, the initial condition is readily obtained, just putting $t=0$ in this equation,

$$f_{1i} = f_1(t=0) = n_0 \frac{q\Phi_1}{m} \frac{1}{2i\pi(v-\tilde{v}_\varphi)} \left[i\pi \frac{f_0'(v)}{n_0} - D_p(v) \right]. \quad (31)$$

We can see that the initial condition of this solution is a function $f_{1i}(v)$ which has, as the subsequent solution for $t>0$, a remarkable mathematical property; its analytical continuation in the complex v-plane possesses a complex pole at $v=\tilde{v}_\varphi$. This pole does not correspond to any unphysical property since there is no longer a singularity for real velocities; it merely leads to some widened peak (modulated in space by the common e^{ikx} factor), all the less marked that the imaginary part (the damping) is large. But distribution functions with or without poles appear to have quite different behaviors. This result is more striking when stated in the reverse way; when the system is initialized with a perturbed

distribution function which has a complex pole $\tilde{v}_0$, its subsequent evolution is partly determined by this complex pole. The macroscopic variables as ϕ_1 then always contains a part which evolves in time as $e^{-ik\tilde{v}_0 t}$, the phase velocity and damping of this part being predetermined from the initial condition. This provides an easy means for building the infinity of complex solutions, but one must realize that this solution corresponds to a very peculiar initial condition: To get it, one must respect the above form in modulus and phase for each value of x. The system can indeed be prepared in such a way in the simulation (see next section), but from arbitrary initial conditions, there is no chance that any of these solutions get excited.

What happens when the initial distribution does not include any complex pole? The temporal evolution must then be imposed by some "generic" phase mixing, instead of being predetermined by special phase relations in the initial condition. And when does the initial condition of one of our solutions not include any complex pole? When the numerator of Eq. (31) is zero for $v=\tilde{v}_\varphi$, which means

$$D(\tilde{v}_\varphi) = D_p(\tilde{v}_\varphi) - i\pi f_0'(\tilde{v}_\varphi)/n_0 = 0. \tag{32}$$

To make it short, we will refer hereafter to this condition that there is no pole in the lower half plane of the complex velocity plane as the "no pole" condition (poles in the upper half plane have no consequence for the considered decreasing wave packets). Let us outline that the poles that we are speaking about are the poles of the distribution function when the variable v is continued in the complex plane. They must not be confused with the poles involved in Landau's theory, which are the poles in ω of the integrand of the inverse Laplace transform, i.e., the zeros of the dispersion relation.

Equation (32) shows that the "no pole" condition directly leads to the well-known kinetic dispersion equation first derived by Landau. It is thus the condition that allows for the choice made among the infinite number of complex solutions. Where does it come from? The reason is just a statistical one: All other solutions are unlikely because, as the pole exists identically at all times, it should be injected in the initial condition, which cannot occur by chance, as emphasized above. Let us outline that this "no pole" condition actually exists, consistently, in Landau's theory. But it is never highlighted nor commented; it is more generally understood as some mathematical precaution for integrating in the complex plane and not as a basic physical ingredient. From the viewpoint of the particle signature of linear Landau damping, the present calculation fully explains what is observed in simulations; the kinetic Landau solution is the only solution that avoids the existence of a pole. In the case of a weak damping $|\gamma| \ll \omega_p$, Eq. (27) means that all the solutions would display a finite peak centered at the resonant velocity (and modulated in x), except the kinetic solution with Landau damping, which does not present anything particular in this vicinity. This result is at odds with the usual interpretation in terms of resonant particles, which leads us to think that Landau damping is due to a strong perturbation in the distribution function at the resonant velocity. Let us nevertheless outline that it does not mean that the particles with a resonant velocity does not have a particular role in the dissipation mechanism.

V. INITIAL CONDITIONS THAT KILL LANDAU DAMPING

While Landau's solution appears to be the most likely solution in terms of initial conditions, all the infinity of other conditions can also be easily obtained in a simulation (and may be not impossible in laboratory experiments). One has just to choose an initial condition with an arbitrary complex pole, and check that the decrease is then fixed, not by Landau damping, but by the imaginary part of the pole injected. We are not aware of such numerical results in the literature and show here the first demonstration of it. It is worth noticing however that effects such as "plasma echoes"[30] have already been known for long to overstep Landau's damping, which is of course closely related, even if nonlinear.

The simulations presented in this section are set with $|\gamma_0| < |\gamma_L|$ ($\gamma_0=-0.005$ and $\gamma_0=-0.02$). They are initialized with a distribution function perturbation close to the one used in LDR simulation, but a term with a pole in the complex plane is added,

$$\delta F(v,x,t=0) = \mathrm{Re}\left[N_1(x)\left(\alpha + \frac{\beta}{v-\tilde{v}_0}\right)\right]\frac{1}{\sqrt{2\pi}v_t} \times \exp\left\{-\frac{[v-V_1(x)]^2}{2v_t^2}\right\}, \tag{33}$$

where $\tilde{v}_0=(\omega_L+i\gamma_0)/k$, and now $N_1(x)=N_1 \exp i(kx+\phi)$ where N_1 is a real number. In order to study the influence of such a pole on the wave damping only, ω_L is chosen to be equal to the real part of the exact solution of Eq. (32), and γ_0 is set freely (the index 0 characterizes the pole). To avoid any confusion with nonlinear effects near the resonant velocity, these simulations are conducted with the linearized version of the code. In the numerical illustration shown, we have set $\alpha=1$ and $\beta=0.5\gamma_0/k$. This choice is largely arbitrary, but it allows us to preserve a positive distribution function.

Contrary to LDR simulation, because of the term added with a pole, the density and the mean velocity of the perturbed distribution function can be notably different from the given functions $N_1(x)$ and $V_1(x)$. This would make more delicate the initialization of the system if we wanted to exactly obtain only the forward Langmuir mode. We did not really try here to solve this problem [the form of Eq. (33) does not correspond to the theoretical form derived in the previous section]. The functions $N_1(x)$ and $V_1(x)$ have been chosen relatively arbitrarily, and we will see that the asymptotic behavior is indeed purely determined by the given complex pole and is therefore independent of this choice.

In simulation NLD1, the pole has a frequency $\omega_L+i\gamma_0$, with $\gamma_0=-0.005>\gamma_L$, and $\beta=0.47\times10^{-3}$. The functions $N_1(x)$ and $V_1(x)$ simply correspond to $N_{1,0}(x)$ and $V_{1,0}(x)$ [see Eqs. (9) and (10)], as in LDR simulation, i.e., derived from the cold plasma relations, Eqs. (9) and (10). This choice is the simplest, but it corresponds to initial density and the

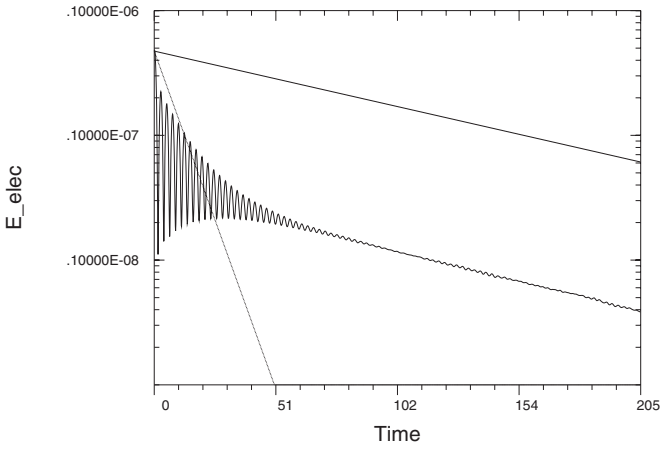

FIG. 6. Simulation NLD1 (with a pole). Time evolution of the electric energy as a function of time, for initial conditions with a pole at frequency $\omega_L+i\gamma_0$, with $\gamma_0=-0.005>\gamma_L$. Curve: Simulation result (logarithmic scale). Steeper straight line: Value given by the theoretical Landau damping coefficient $\gamma_L=-0.0609$ (the slope is $2\gamma_L$). Less steep straight line: Theoretical value given by imaginary part of the pole $\tilde{v}_0$ (the slope is $2\gamma_0$).

velocity n_1 and v_1 which are then quite different for the cold plasma values $N_{1,0}(x)$ and $V_{1,0}(x)$, so that waves in both senses are initially excited as well.

To make the comparison easier with LDR simulation, in particular for comparing the phase space figures (see hereafter), a second initialization has been done (simulation NLD2), where the parameters have been chosen heuristically to be somewhat closer to a single Langmuir mode, and therefore to $n_1=N_{1,0}$ and $v_1=V_{1,0}$. This result has been approximately obtained by choosing $V_1(x)=10\ V_{1,0}(x)$ and $N_1(x)=0.1\ N_{1,0}(x)$. In NLD2, $\omega_L+i\gamma_0$, with $\gamma_0=-0.02>\gamma_L$ and $\beta=0.30\times10^{-2}$. Once again, initializing with the exact form given in the theoretical section is certainly possible, but more difficult. This is left for future works.

The evolution of the electric energy is shown in Fig. 6 for NLD1 and Fig. 7 for NLD2. The less oblique straight line

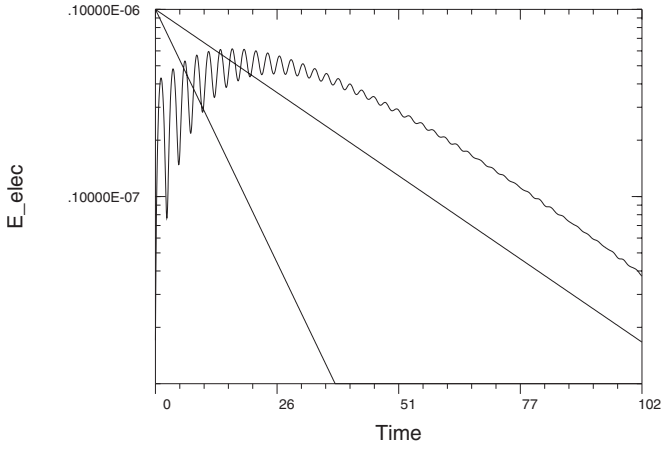

FIG. 7. Simulation NLD2 (with a pole). Time evolution of the electric energy as a function of time, for initial conditions with a pole at frequency $\omega_L+i\gamma_0$, with $\gamma_0=-0.02>\gamma_L$. Black curve: Simulation result (logarithmic scale). Steeper straight line: Value given by the theoretical Landau damping coefficient $\gamma_L=-0.0609$ (the slope is $2\gamma_L$). Less steep straight line: Theoretical value given by the imaginary part of the pole $\tilde{v}_0$ (the slope is $2\gamma_0$).

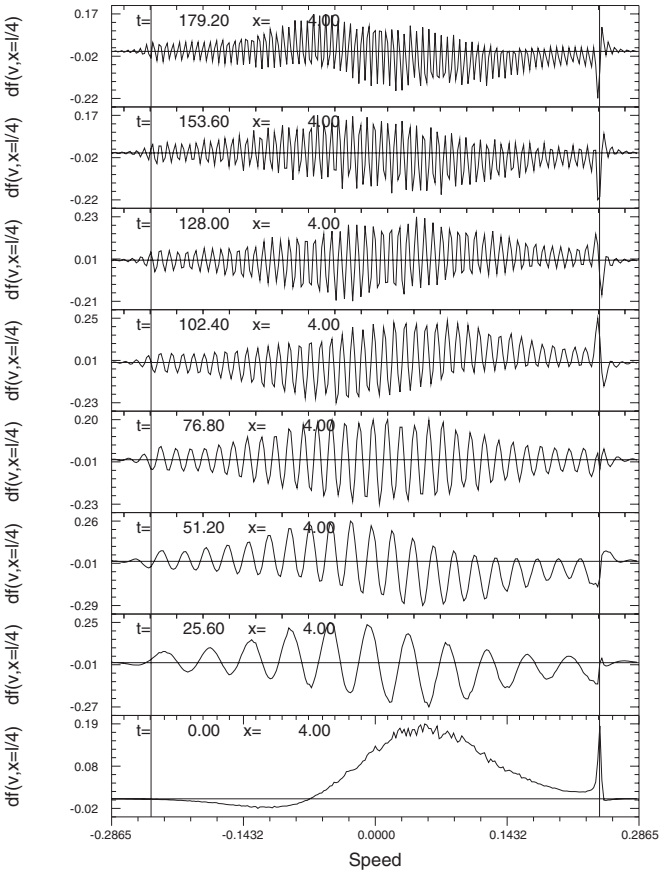

FIG. 8. Simulation NLD1 (as in Fig. 6). Perturbation $f_1(v,x_0,t)$ at various times for the simulation with initial conditions with a pole at frequency $\omega_L+i\gamma_0$, with $\gamma_0=-0.005$.

corresponds to the damping rate γ_0 imposed by the pole set in the initial conditions, and the other straight line corresponds to Landau damping. A transition regime can be observed, due to the initial conditions which does not correspond exactly to the eigenmode, this transition being different in NLD1 and NLD2. It is however evident that, after this transition regime, the damping is the same in the two simulations. It is ruled by γ_0, not by other details of the initial conditions, and *not by the Landau rate*.

The distribution functions $f_1(v,x_0,t)$, at a fixed position and at various times, are displayed in Fig. 8 for NLD1 and Fig. 9 for NLD2. Unlike Landau damping, we can notice a particular feature in the distribution function for velocities close to the resonant velocity. This feature was set in the initial perturbation and is preserved for $t>0$. When the damping imposed by the pole $\tilde{v}_0$ in the initial distribution function is larger than the Landau damping rate, $|\gamma_0|>|\gamma_L|$, the observable damping rate of the wave is approximately the weakest, i.e., the Landau damping rate, as confirmed in Fig. 10. The reason is, once again, that our initial condition given by Eq. (33) does not correspond to a pure eigenmode and that the two effects, Landau and non-Landau damping are competing. If initializing with the theoretical form of $f_{1i}(v)$ given by Eq. (31), we should exactly find the strong damping. The increase of the electric energy in the early times of the simulation also shows evidence that we are not

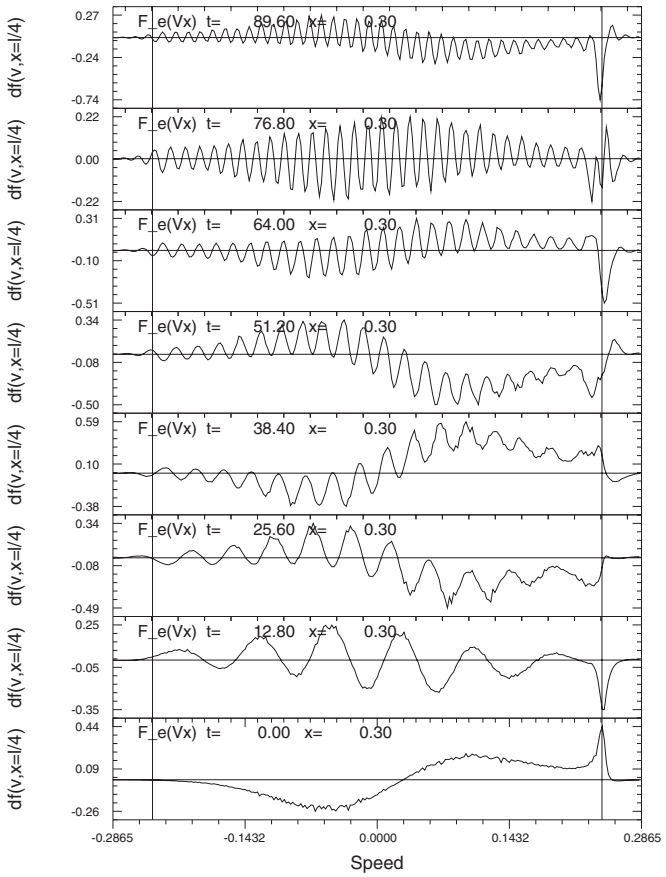

FIG. 9. Simulation NLD2 (as in Fig. 7). Perturbation $f_1(v,x_0,t)$ at various times for the simulation with initial conditions with a pole at frequency $\omega_L+i\gamma_0$, with $\gamma_0=-0.02$.

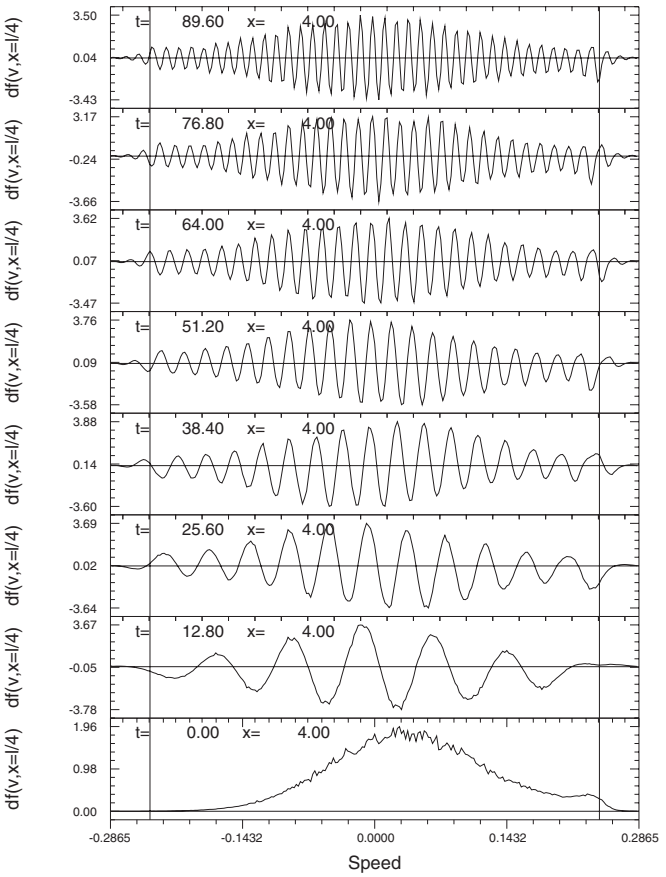

FIG. 11. Perturbation $f_1(v,x_0,t)$ at various times for the simulation with initial conditions with a pole at frequency $\omega_L+i\gamma_0$, with $\gamma_0=-0.08$.

very close to the eigenmode. The difference does not come only, in this case, from the presence of a backward wave (which only explains the modulation), but also from the very characteristics of the mode itself. When inserting the pole as we do, we clearly put too much free energy in the particle distribution, as compared to the initial electric energy. However, as shown previously, a particular feature near the resonant velocity is still visible, at any time, as shown in Fig. 11. Finally, a visual hint of what is going on can be obtained by looking to phase space evolution. Figures 5 and 12 show the phase spaces at $t=0$ and $t=12.8$ and $t=25.6$ for the Landau damping reference simulation (LDR) and non-Landau damping (NLD2). One can understand first why the initial condition in Fig. 12 is less probable; the pole, whose signature can be seen on the upper part of the phase space, implies not only a notable peak in the distribution, but with a sign and an amplitude which depends on x in a very special manner (sinusoidal in the present monochromatic case).

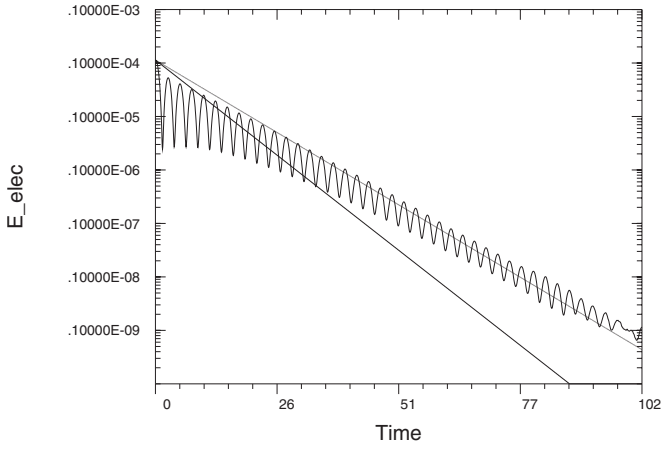

FIG. 10. Same as Fig. 6, for initial conditions with a pole at frequency $\omega_L+i\gamma_0$, with $\gamma_0=-0.08<\gamma_L$ ($\alpha=1$ and $\beta=0.76\times10^{-2}$).

VI. DISCUSSION AND CONCLUSION

The present paper studies the plasma evolution following an initial perturbation, depending on the shape of the initial perturbed distribution function $\delta f(v,x)$. It shows that, in agreement with Landau's pioneering paper,[1] the classical Landau damping occurs only when the initial perturbation is an entire function of v. This result is obtained here using both an analytical calculation of the varying distribution function and a low noise numerical simulation. These approaches allow further investigation of the cases when the initial condition does possess poles in the complex v plane. It is worth mentioning that Akhiezer *et al.*,[10] using the classical Landau calculation, outlined this problem more than 30 years ago, even exhibiting two examples with asymptotic behaviors different from Landau's. It seems however that these conclusions are often forgotten nowadays; they are never evoked in particular when analyzing the long time behavior

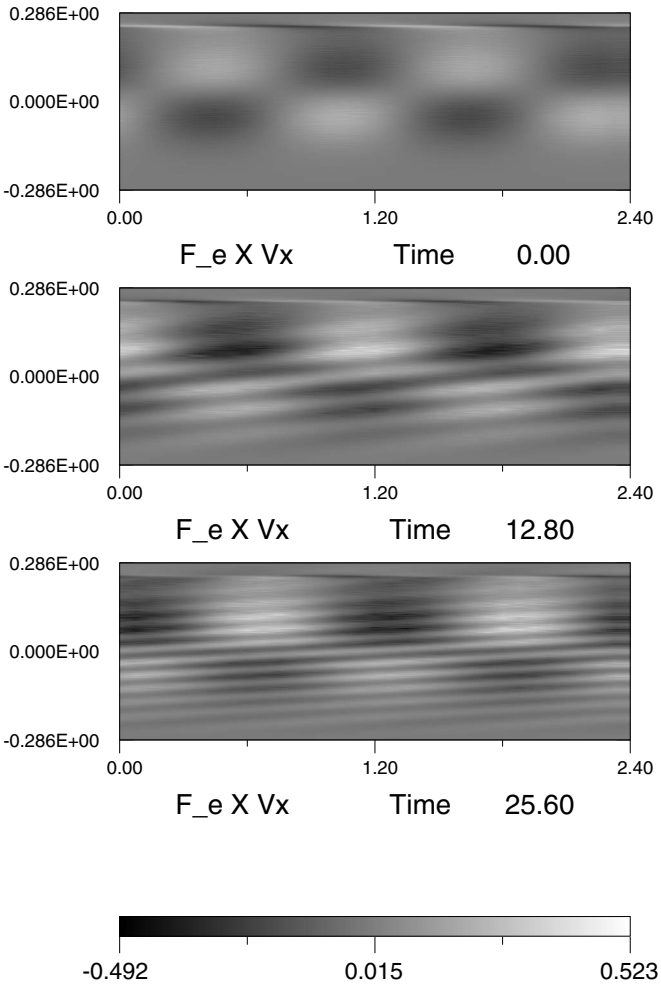

FIG. 12. Simulation NLD2. Electron phase space at various times at the beginning of the simulation with a pole.

of the nonlinear Landau damping. The specificity of the present work consists of searching systematically the "eigenmodes" of the system, i.e., the modes for which the macroscopic quantities vary as complex exponentials. We show that any real frequency and damping rate can be obtained by choosing adequate (but still regular) initial perturbations. It must be outlined that this behavior is not transient; it concerns the asymptotic behavior (in the linear limit). Nevertheless, Landau's solution can be told to be "universal" in the sense that it is the most probable asymptotic behavior; all solutions tend toward it at the condition that one starts with an initial perturbation $\delta f(v,x)$ which does not contain any complex pole (when analytically continued in the lower half complex plane), which will never get forgotten. This condition is most generally verified for ordinary initial conditions because a complex pole corresponds to very special phase relations between the different points in space. But it is a condition that may have to be taken into account when interpreting nonlinear results, in theory or in simulation, because the nonlinear effects could change the mathematical properties of $\delta f(v,x)$ and reorganize the phases. The nonlinear time when these effects could occur, which of course depends on the amplitude of the perturbation, has to be compared to the time when Landau's solution emerges as the linear asymptotic solution, which depends on the shape of $\delta f(v,x)$. This point seems to have been disregarded hitherto.

In Sec. III, thanks to numerical simulations, we have shown that Landau damping occurs *although* the distribution function does not display any particular feature at the resonant velocity. In Sec. IV, looking back to theory, we have shown furthermore that Landau damping actually occurs *because* nothing apparent happens at the resonant velocity; the presence of a pole would always correspond to some notable feature at the resonant velocity and it is actually the "no pole" condition which leads to the dispersion equation of the classical Landau's damped kinetic modes. It must be emphasized anyway that the initialization with a pole, even if ad hoc, implies perfectly regular distribution functions, invalidating the usual belief that solutions different from Landau correspond to "pathological" distribution functions. The question of inverse Landau damping has not been treated in this paper. Approaching it by similar methods may be the subject of future works.

In Ref. 17, looking for undamped solutions (and therefore different from Landau's one), the authors have rightly outlined that "linear theory is, in general, incapable of describing undamped waves of any amplitude, no matter how small or large, with smooth distribution functions." We confirm this result but have demonstrated, however, that linear theory is actually able to describe solutions of any damping rate (except zero but possibly much less than the classical Landau's rate), which correspond to perfectly regular distributions. Even for the case $\gamma=0$, one can consider that, in the small amplitude limit, the nonlinear effects have only the minor role of regularizing the distribution function in the close vicinity of the resonant velocity, but that they are without effect on the main physical mechanisms that rule the wave propagation and damping, which are basically independent of this regularization and therefore only depend on the linear effects.

The calculations of Landau damping that use energy-based arguments, in spite of the insight into the nonlinear properties of the waves that they demand, should not lead to the wrong conclusion that, contrary to Landau's calculation, the linear theory is not self-sufficient and actually conceals nonlinear effects. In particular, the linear concept of "resonant particles" must of course not be confused with the nonlinear one of "trapped particles." This fact has indeed been shown in recent laboratory experiments.[31] Quasilinear effects can lead in some circumstances (when starting from a spectrum of large amplitude waves) to the formation of a plateau in some velocity range in resonance with the waves (e.g., Ref. 32). This effect is then concomitant with the wave damping, but one must not confuse either of the two phenomena: The quasilinear diffusion is not the cause of the linear damping and the width of the plateau (which depends on the wave amplitudes and on the width of the assumed wave spectrum) does not correspond to the resonant range of linear theory, even if both occur to be named "resonant range" in the literature. It is actually easy to show that nonlinear effects have a very minor role in the results presented in this paper. When looking to the perturbed distribution function δf averaged in x on the whole simulation box, ev-

erything cancels to zero except a small remnant close to the resonant velocity when using the standard version of the perturbative PIC code. When using instead the linearized version of it, this remnant disappears, as expected, while the damping observed in the electric energy decrease does not change at all.

The present results seriously question the common view of the role of "resonant particles" and suggest a different understanding of it. The notion of resonant particles is mainly based on the pioneer energy calculation made by Dawson.[4] It will be useful to provide a comparable calculation, complete enough to include the present results and to allow us to interpret them from the point of view based on energy computations. This work is in progress and will be presented in a forthcoming paper. It should allow us to clarify the relations between the present "no pole" condition and the usual, but ill-defined, condition of "forgetting initial conditions."

The role of the "ballistic effects," rarely emphasized in textbooks, appears more evident in the present numerical and theoretical results. We have shown that this folding of phase space, which has been outlined in a number of nonlinear studies (e.g., Ref. 16) is indeed always present, from the linear stage, and not limited to resonant particles.

As an important by-product, the present paper is expected to change the physical intuition about Landau damping. The need of such an intuition is of course evident whenever one tries to teach or popularize Landau damping without going into the details of calculation. As pointed out by Refs. 8 and 9, some of the most common analogies (cf. "surfer analogy") are clearly too simplistic to provide the right image of the physics at play. However, no alternative simple idea has been proposed hitherto; the respective roles of energy exchanges and pure phase mixing remain to be clearly explained. Such a physical intuition is also necessary whenever one tries to get a physical guess of different kinetic phenomena. An example is the mirror instability, which is another typical case of wave-particle resonance problem. In Ref. 33, the authors compare the growth rate of the instability with an estimation of the "number of resonant particles" supposedly responsible for the growth. They find the paradoxical result that the first is inversely proportional to the second, which puts some doubts on the physical interpretation. The present calculations show that the solution of this paradox is not likely to be found in the deformation of f_1 around the resonant velocity. Shedding some light on these questions should therefore help, not only for pedagogy but also in quite active areas of research.

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APPENDIX: DISTRIBUTION FUNCTION FOR A WAVE PACKET

Once the expression of f_1 is known for the real eigenmodes, as given by Eq. (23), its expression for the complex eigenmodes is straightforwardly obtained by the following integration over v_φ:

$$f_1 = n_0 \frac{q\Phi_1}{m} \frac{1}{-2\pi i} \int_{-\infty}^{+\infty} e^{-ikv_\varphi t} \frac{dv_\varphi}{v_\varphi - \tilde{v}_\varphi} \left[P\left\{ \frac{f_0'(v)/n_0}{v - v_\varphi} \right\} + D_p(v_\varphi)\,\delta(v - v_\varphi) \right]. \tag{A1}$$

The second term, with a Dirac distribution, is easy to integrate and provides

$$n_0 \frac{q\Phi_1}{m} \frac{1}{-2\pi i} \frac{1}{(v - \tilde{v}_\varphi)} D_p(v) e^{-ikvt}. \tag{A2}$$

The first term is the principal value of the integral,

$$\frac{q\Phi_1}{m} \frac{f_0'(v)}{2i\pi} P \int_{-\infty}^{+\infty} \frac{e^{-ikv_\varphi t}}{(v_\varphi - v)(v_\varphi - \tilde{v}_\varphi)} dv_\varphi, \tag{A3}$$

which can be decomposed as the sum,

$$-\frac{q\Phi_1}{m} \frac{1}{2i\pi} \frac{f_0'(v)}{v - \tilde{v}_\varphi} \left[P \int_{-\infty}^{+\infty} \frac{e^{-ikv_\varphi t}}{v_\varphi - \tilde{v}_\varphi} dv_\varphi - P \int_{-\infty}^{+\infty} \frac{e^{-ikv_\varphi t}}{v_\varphi - v} dv_\varphi \right]. \tag{A4}$$

The two terms of this sum are classical inverse Laplace integrals, except that the principal value introduces a factor $1/2$ in the second one because its pole is real,

$$\frac{q\Phi_1}{m} \frac{f_0'(v)}{v - \tilde{v}_\varphi} \left(e^{-ik\bar{v}_\varphi t} - \frac{1}{2} e^{-ikvt} \right). \tag{A5}$$

It is worth noticing that the two integrals can be calculated via a contour deformation, exactly as in Landau's theory. We tried to make clear here that this is just a mathematical knack, independent from the physical arguments given in the text. Finally, the two contributions [Eqs. (A2) and (A5)], when added, provide the expression given in Eq. (27).

[1] L. D. Landau, J. Phys. (USSR) **10**, 25 (1946).
[2] J. H. Malmberg and C. B. Wharton, Phys. Rev. Lett. **17**, 175 (1966).
[3] D. D. Ryutov, Plasma Phys. Controlled Fusion **41**, A1 (1999).
[4] J. Dawson, Phys. Fluids **4**, 869 (1961).
[5] A. Kildal, Nuovo Cimento **20**, 104 (1961).
[6] A. Kildal, J. Nucl. Energy, Part C **3**, 256 (1961).
[7] W. E. Drummond, Phys. Plasmas **11**, 552 (2004).
[8] D. F. Escande, S. Zekri, and Y. Elskens, Phys. Plasmas **3**, 3534 (1996).
[9] Y. Elskens and D. Escande, *Microscopic Dynamics of Plasmas and Chaos* (Institute of Physics, Bristol, 2003).
[10] A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Sitenko, and K. N. Stepanov, *Plasma Electrodynamics* (Pergamon, Oxford, 1975).
[11] J. Buchner, C. Dum, and M. Scholer, *Space Science Simulation* (Springer-Verlag, Heidelberg, 2003).
[12] N. G. van Kampen and B. U. Felderhof, *Theoretical Methods in Plasma Physics* (North-Holland, Amsterdam, 1967).
[13] K. M. Case, Ann. Phys. **7**, 349 (1959).
[14] M. C. Firpo and F. Doveil, Phys. Rev. E **65**, 016411 (2001).
[15] J. Li and G. O. Spies, Phys. Fluids B **3**, 1158 (1991).
[16] T. O'Neil, Phys. Fluids **8**, 2255 (1965).

[17]J. P. Holloway and J. J. Dorning, Phys. Rev. A **44**, 3856 (1991).
[18]M. Brunetti, F. Califano, and F. Pegoraro, Phys. Rev. E **62**, 4109 (2000).
[19]M. B. Isichenko, Phys. Rev. Lett. **78**, 2369 (1997).
[20]G. Manfredi, Phys. Rev. Lett. **79**, 2815 (1997).
[21]C. Lancelloti and J. J. Dorning, Phys. Rev. Lett. **81**, 5137 (1998).
[22]A. V. Ivanov, I. H. Cairns, and P. A. Robinson, Phys. Plasmas **11**, 4649 (2004).
[23]F. Valentini, V. Carbone, and P. L. Veltri, Phys. Rev. E **71**, 017402 (2005).
[24]M. C. Firpo, F. Doveil, Y. Elskens, P. Bertrand, M. Poleni, and D. Guyomarc'h, Phys. Rev. E **64**, 026407 (2001).
[25]R. De Marco, V. Carbone, and P. L. Veltri, Phys. Rev. Lett. **96**, 125003 (2006).
[26]R. De Marco, V. Carbone, and P. L. Veltri, J. Comp. Physiol. **107**, 309 (1993).
[27]S. E. Parker and W. W. Lee, Phys. Fluids B **5**, 77 (1993).
[28]R. D. Sydora, *Space Plasma Simulation*, Lecture Notes in Physics (Springer, Berlin, 2003), Vol. 615, pp. 109–124.
[29]I. M. Guelfand and G. E. Chilov, *Les Distributions* (Dunod, Paris, 1962).
[30]E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics*, Course of Theoretical Physics (Pergamon, Oxford, 1981), Vol. 10.
[31]F. Doveil, D. F. Escande, and A. Macor, Phys. Rev. Lett. **94**, 085003 (2005).
[32]G. Laval and R. Pellat, *Plasma Physics*, Les Houches Series, Université de Grenoble, Summer School of Theoretical Physics, edited by C. DeWitt and J. Peyraud (Gordon and Breach, New York, 1975), pp. 219–291.
[33]D. J. Southwood and M. G. Kivelson, J. Geophys. Res., [Space Phys.] **98**, 9181, DOI: 10.1029/92JA02837 (1993).