Mesoscopic BCS treatment of neutron superfluidity in solid star crust

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Abstract. At a mesoscopic level (large compared with the solid lattice spaceing but small compared with the vortex separation) the relative current density n^i of conduction neutrons in a neutron star crust beyond the neutron drip threshold can be expected to be related to the corresponding particle momentum covector p_i by a linear relation of the form $n^i = \mathcal{K}^{ij} p_i$ in terms of a physically well defined mobility tensor $\mathcal{K}^{ij}$. This effect is describable as an "entrainment" whose effect – wherever the crust lattice is isotropic – will simply be to change the ordinary neutron mass m to an effective mass $m^\star$ such that in terms of the relevant number density n of unconfined neutrons we shall have $\mathcal{K}^{ij} = (n/m^\star)\gamma^{ij}$.

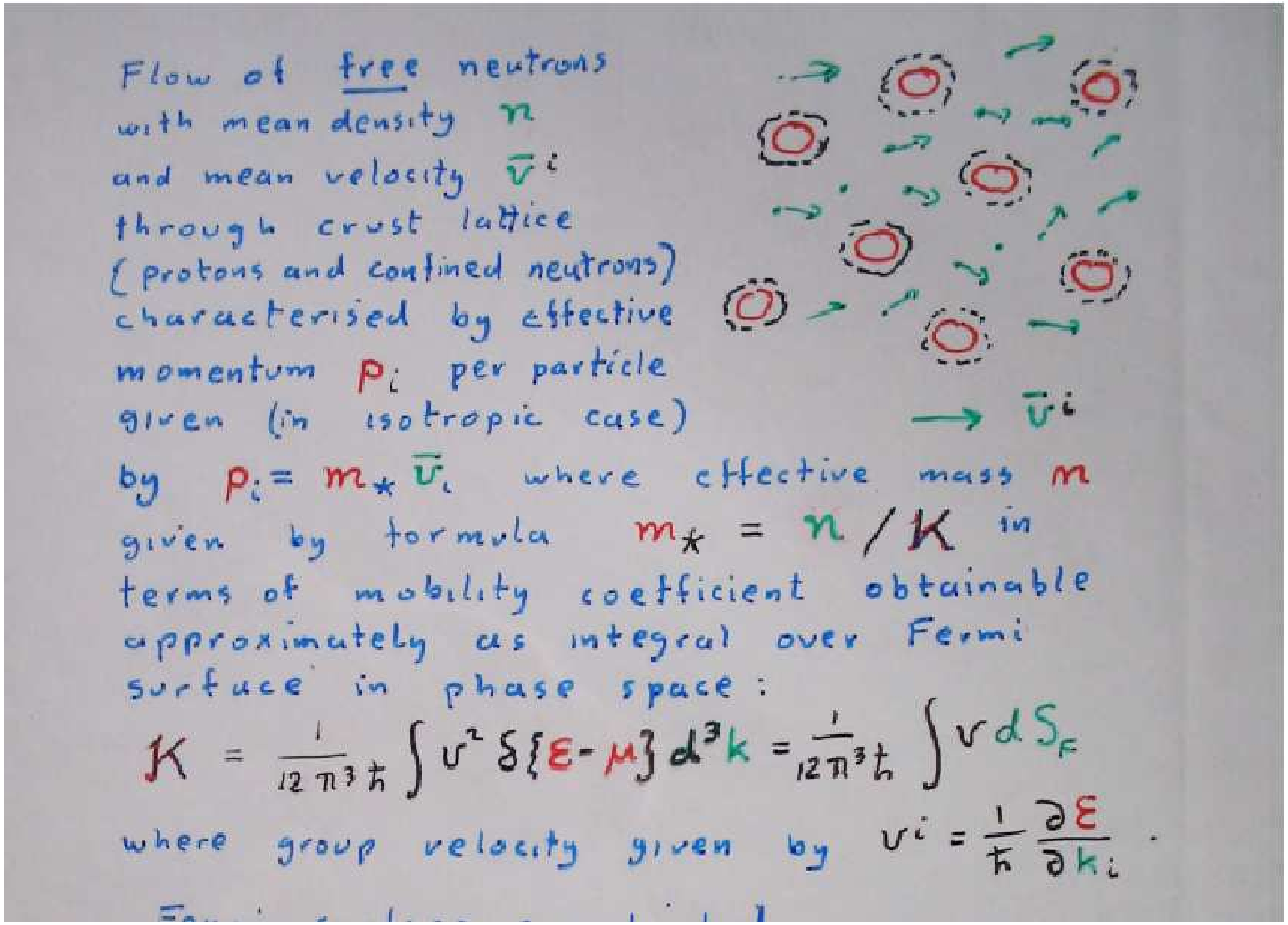

FIG. 1 – The (mesoscopic) effective mass

In a preceding quantum mechanical analysis [1] based on a Hartree type independent particle treatment using Bloch type boundary conditions to obtain the distribution of energy $\mathcal{E}_k$ and associated group velocity ${v_k}^i = \partial\mathcal{E}/\partial k_i$ as a function of wavenumber k_i , it was shown that the mobility tensor would be proportional to a space space volume integral : $\mathcal{K}^{ij} \propto \int 2{v_k}^i{v_k}^j\,\delta\{\mathcal{E}_k - \mu\}$, where μ is the Fermi energy. Using the approach due to Bogoliubov, is shown here [2] that the effect of BCS pairing with a superfluid gap energy density Δ and corresponding pseudoparticle energy function $\epsilon_k = \sqrt{(\mathcal{E}_k - \mu)^2 + \Delta^2}$ will just be to replace the Dirac distributional integrand by the smoother distribution in the formula $\mathcal{K}^{ij} \propto \int {v_k}^i{v_k}^j\,\Delta^2/{\epsilon_k}^3$. It is concluded that the prediction of a large effective mass enhancement will not be significantly effected by this pairing mechanism.

1 Introduction

In the "outer crust" of a neutron star the neutrons will be entirely confined (together with the protons) inside atomic nucle as in ordinary terrestrial matter, but above the "drip" threshold (at about 10^{11} gm/cm^3) a certain fraction of the neutrons in the "inner crust" (and all the neutrons in the deeper layers at densities exceeding the typical nuclear value of the order of 10^{11} gm/cm^3) will be free to form an independently moving current. There is general agreement that except in very young neutron stars for which the temperature may be too high (of the order of 10^9 degrees Kelvin or more) such neutron currents should be able to survive over macroscopically long timescales due to the suppression of (resistive and viscous) dissipation by the effect of superfluid pairing.

Until now, quantum theoretical analysis of the mechanism of neutron superfluidity has mainly concentrated on static configurations, meaning states for which no current is actually flowing relative to the relevant background which, in the crust of a neutron star, will be constituted by the ionic nuclei to which some of the neutrons and all the protons will be confined. Even at densities substantially beyond the neutron drip threshold it should still be possible to obtain a reasonably accurate description for the static case by using an approximation [3] that treats the neighbourhood of each ionic nucleus as if it were isolated in a (roughly spherical) Wigner Seitz type cell. However for the treatment of more general – stationary but non static – configurations involving relative conduction currents it is absolutely necessary to use a more realistic description in which the artificial Wigner Seitz type boundary conditions are replaced by the natural Bloch type periodicity conditions that would be desirable for higher accuracy even in the static case.

The use of appropriate Bloch type periodicity conditions is routine in terrestrial solid state physics, but has so far been applied to neutron star matter only in a simplified Hartree type treatment [1] (of the kind appropriate for the relatively high temperatures expected in very young neutron stars) for which the neutrons are considered to move as independent particles without allowance for the pairing interactions responsible for the superfluid energy gap that (in cool mature neutron stars) allows the currents to persist.

A simplified treatment of this kind has been used to show that the middle layers of a neutron star crust will be characterised by a very low value for the relevant mobility tensor in the formula $n^i = \mathcal{K}^{ij} p_j$ for the current $n^i = n\,\bar{v}^i$ of unbound neutrons (which will be present above the "drip" density of the order of 10^{11} gm/cm^3) with number density n, mean velocity $\bar{v}^i$ and momentum per neutron p_i.

In the independent particle treatment the mobility tensor was shown [1] to be given by a volume integral over the space of Bloch momentum covectors k_i that is expressible in terms of a Dirac distribution with support confined to the Fermi surface – where the relevant energy function $\mathcal{E}_k$ is equal to the chemical potential μ – in the form

$$\mathcal{K}^{ij} = \frac{1}{8\pi^3} \int v_k{}^i v_k{}^j \, \delta\{\mathcal{E}_k - \mu\} \, \mathrm{d}^3 k \,, \qquad (1)$$

in which the relevant group velocity distribution is given by the usual formula

$$v_k^i = \frac{\partial \mathcal{E}_k}{\partial k_i} \,. \qquad (2)$$

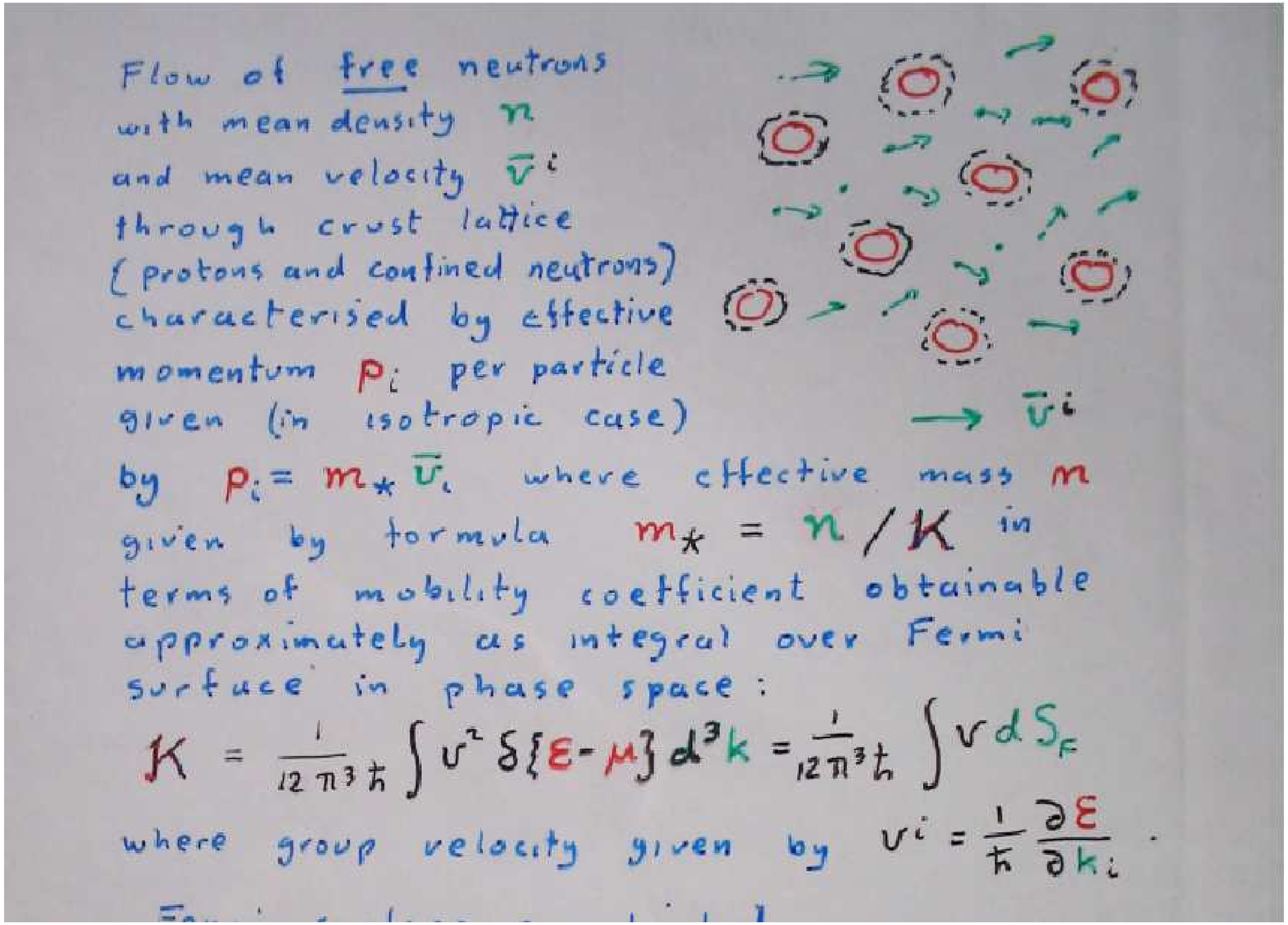

FIG. 2 – The mobility coefficient

The purpose of our work here is to show how the preceding independent particle treatment can be generalised to allow for BCS type pairing using an approach of the kind pioneered by Bogoliubov. One of the main motivations for this is to check the robustness of the conclusions obtained from the simple treatment described above, particularly the prediction of a very low value for the mobility tensor, which is interpretable as meaning that the corresponding effective mass $m^\star = n/3\mathcal{K}^i{}_i$ will become very large compared with the ordinary neutron mass.

Our conclusion is that as a first step towards a more accurate treatment, in cases for which the superfluid pairing can be characterised just by a gap parameter Δ_k , the relevant integral over the Fermi surface will need to be replaced by an phase space volume integral.

This will be given in terms of the pseudo-particle energy

$$\epsilon_k = \sqrt{(\mathcal{E}_k - \mu)^2 + \Delta_k{}^2} \tag{3}$$

by the new formula

$$\mathcal{K}^{ij} = \int v_k{}^i v_k{}^j \frac{\Delta_k{}^2}{(2\pi\epsilon_k)^3}\, \mathrm{d}^3 k\,, \tag{4}$$

in which the expression for the group velocity $v_k{}^\mu$ is the same as in the absence of the gap. It is the diminution of this group velocity that is responsible for the enhancement of the effective mass, which on average should therefore not be greatly affected by the phase space smearing effect produced by the superfluid pairing.

2 Independent particle Hamiltonian

We start on the basis of a second quantised formalism in terms of local Fermionic field annihilation and creation operators $\hat{\psi}$ and $\hat{\psi}^{\dagger}$ depending on space position coordinates r^i taking discrete values on a mesh that is fine compared with the physically relevant lengthscales, and on a spin varible σ taking values $\uparrow$ and $\downarrow$ subject to the usual anticommutation rules

$$[\hat{\psi}_{\sigma}\{\vec{r}\}, \hat{\psi}_{\sigma'}\{\vec{r}'\}]_{+} = 0\,, \qquad [\hat{\psi}^{\dagger}{}_{\sigma}\{\vec{r}\}, \hat{\psi}^{\dagger}{}_{\sigma'}\{\vec{r}'\}]_{+} = 0\,, \tag{5}$$

$$[\hat{\psi}^{\dagger}{}_{\sigma}\{\vec{r}\}, \hat{\psi}_{\sigma'}\{\vec{r}'\}]_{+} = \delta_{\sigma\sigma'}\delta\{\vec{r}, \vec{r}'\}\,. \tag{6}$$

We use a quadratic Hamiltonian operator of the form

$$\hat{H} = \sum_{\vec{r}} \left(\hat{\mathcal{H}}_{\text{kin}}\{\vec{r}\} + \hat{\mathcal{H}}_{\text{pot}}\{\vec{r}\} + \hat{\mathcal{H}}_{\text{int}}\{\vec{r}\} \right), \tag{7}$$

in which the last term $\hat{\mathcal{H}}_{\text{int}}\{\vec{r}\}$ will be absent in the independent particle limit corresponding to the kind of model used [1] in our preceeding first quantised treatment.

In this independent particle limit, only the first two contributions are present, of which the simplest is given by

$$\hat{\mathcal{H}}_{\text{pot}}\{\vec{r}\} = V\{\vec{r}\} \sum_{\sigma} \hat{\psi}^{\dagger}{}_{\sigma}\{\vec{r}\} \hat{\psi}_{\sigma}\{\vec{r}\} \tag{8}$$

where $V\{\vec{r}\}$ is a position dependent potential.

Iin a "self consistent" model the potential should itself be given in the relevant (ground state or conducting) reference state $|\rangle$ by a sum over spin values and over relevant particle species – which in the context under consideration means not just protons but also neutrons – of an appropriate (Green function weighted) weighted mean over neighbouring positions of the expected number density, as given for the neutrons with which we are primarily concerned here by

$$n_\sigma\{\vec{r}\} = \langle|\hat{n}_\sigma\{\vec{r}\}|\rangle\,, \qquad \hat{n}_\sigma\{\vec{r}\} = \hat{\psi}^\dagger{}_\sigma\{\vec{r}\}\hat{\psi}_\sigma\{\vec{r}\}\,. \tag{9}$$

The (more or less "self consistent) contribution $V\{\vec{r}\}$ is supposed to represent the averaged effect on the neutrons of the ionic lattice (which in the crudest approximation can be taken to be given just by an array of spherical square wells).

It will be assumed that the potential can be taken to be of periodic crystalline type, with

$$V\{\vec{r}+\ell^a\vec{e}_a\} = V\{\vec{r}\} \tag{10}$$

for any triad of integers ℓ^a $(a=1,2,3)$ in which the lattice basis vectors $\vec{e}_a$ may be interpreted as representing the interionic spacing in the solid case that will be relevant at very lower temperature, but should in principle be taken to be much larger (so as to generate a giant cell interpretable as a typical mesoscopic average over a locally disordered configuration) for applications above the relevant melting temperature, at which it is to be expected that (unlike the weaker electron pairing mechanism in ordinary terrestrial superconductors) the superfluid neutron pairing mechanism will still be intact.

The corresponding formula for the kinetic contribution will be given by

$$\hat{\mathcal{H}}_{\rm kin}\{\vec{r}\} = \sum_\sigma \hat{\psi}^\dagger{}_\sigma\{\vec{r}\}\mathcal{H}_{\rm kin}\hat{\psi}_\sigma\{\vec{r}\} \qquad (11)$$

in which $\mathcal{H}_{\rm kin}$ belongs to is a self adjoint differential operator in the category specified in terms of a gauge covector a_i by an expression of the familiar form

$$\mathcal{H}_a = -\gamma^{ij}(\nabla_i + ia_i)\frac{1}{2m\{\vec{r}\}}(\nabla_i + ia_i) \qquad (12)$$

in which γ^{ij} is the Euclidean space metric and $m\{\vec{r}\}$ is the local effective mass. In a crude approximation this could be taken to have a fixed value close to the ordinary neutron mass value $m_{\rm n}$ but in a more accurate treatment it will need to be given a rather smaller value [3] inside the ionic potential wells.

The covector with components a_i is a gauge field allowing for the possibility of adjustment of the phases of the field operators $\hat{\psi}_\sigma\{\vec{r}\}$. In applications to particles with non zero electric charge (e say) such as the electrons in an ordinary terrestrial superconductor or the protons in the deeper layers of a neutron star, the presence of such a field (taking the form $a_i = eA_i$) would be necessary for the treatment of magnetic effects, but in the uncharged case of the crust neutrons with which we are concerned here it will always be possible to work in the standard gauge for which this covector is simply set to zero, $a_i = 0$, which means that we simply take

$$\mathcal{H}_{\rm kin} = \mathcal{H}_0 \,. \tag{13}$$

The neutron current density operators, will be given, for each value of the spin variable σ , by

$$\hat{n}^{i}_{\sigma}\{\vec{r}\} = \frac{1}{2im\{\vec{r}\}}\gamma^{ij}\left(\hat{\psi}^{\dagger}{}_{\sigma}\{\vec{r}\}\nabla_j\hat{\psi}_{\sigma}\{\vec{r}\} - (\nabla_j\hat{\psi}^{\dagger}{}_{\sigma})\hat{\psi}_{\sigma}\{\vec{r}\}\right) . \tag{14}$$

Our objective of is to evaluate the mean value of this quantity, as given by the corresponding space averaged operator

$$\hat{\bar{n}}^{i}_{\sigma} = \sum_{\vec{r}} \hat{n}^{i}_{\sigma} , \tag{15}$$

as a function of the associated momentum in a stationary state that is non static (and therefore non isotropic, since the mean current will characterise a preferred direction) but uniform over a mesoscopic volume.

3 Use of Bloch eigenfunctions

Subject to the usual Bloch type boundary conditions for a mesoscopic material sample of parallelopiped form – with a unit volume that is taken to be very large compared with the elementary lattice cells under consideration – the Hamiltonian will determine a complete orthonormal set of scalar eigenfunctions $\varphi_k\{\vec{r}\}$ satisfying the the requirement of invariance (subject to the normalisation condition) with respect to infinitesimal variations $\delta\varphi_k\{\vec{r}\}$ of the energy integral

$$\mathcal{E}_k = \sum_{\vec{r}} {\varphi_k}^*\{\vec{r}\}\Big(\mathcal{H}_{\rm kin} + V\{\vec{r}\}\Big)\varphi_k\{\vec{r}\}\,, \qquad (16)$$

(using * to indicate complex conjugation).

The Bloch functions can be labelled by a wave covector k_i such that

$$\varphi_k\{\vec{r}\} = u_k\{\vec{r}\}\,\mathrm{e}^{ik\cdot\vec{r}}\,, \tag{17}$$

using the abbreviation $k \cdot \vec{r} = k_i\, r^i$, where $u_k\{\vec{r}\}$ satisfies the same ordinary lattice periodicity conditions as the lattice potential $V\{\vec{r}\}$, namely

$$u\{\vec{r} + \ell^a \vec{e}_a\} = u\{\vec{r}\}\,. \tag{18}$$

They are characterised by the orthonormality conditions

$$\sum_{\vec{r}} \varphi_k^*\{\vec{r}\}\varphi_{k'}\{\vec{r}\} = \delta_{kk'} \tag{19}$$

as solutions of the elliptic eigenvalue equation

$$\big(\mathcal{H}_0 + V\{\vec{r}\}\big)\varphi_k\{\vec{r}\} = \mathcal{E}_k\varphi_k\{\vec{r}\}\,. \tag{20}$$

Setting the wave number vector k_i in place of a_i in the definition (12) this eigenvalue equation can be usefully rewritten in terms of the ordinarily periodic functions u_k as

$$\left(\mathcal{H}_k + V\{\vec{r}\}\right)u_k\{\vec{r}\} = \mathcal{E}_k u_k\{\vec{r}\}\,. \tag{21}$$

The reality of the potential function implies that the phases of the eigenfunctions can be chosen in such a way that for each wavenumber value

$$\varphi_k^*\{\vec{r}\} = \varphi_{-k}\{\vec{r}\}\,, \qquad u_k^*\{\vec{r}\} = u_{-k}\{\vec{r}\}\,. \tag{22}$$

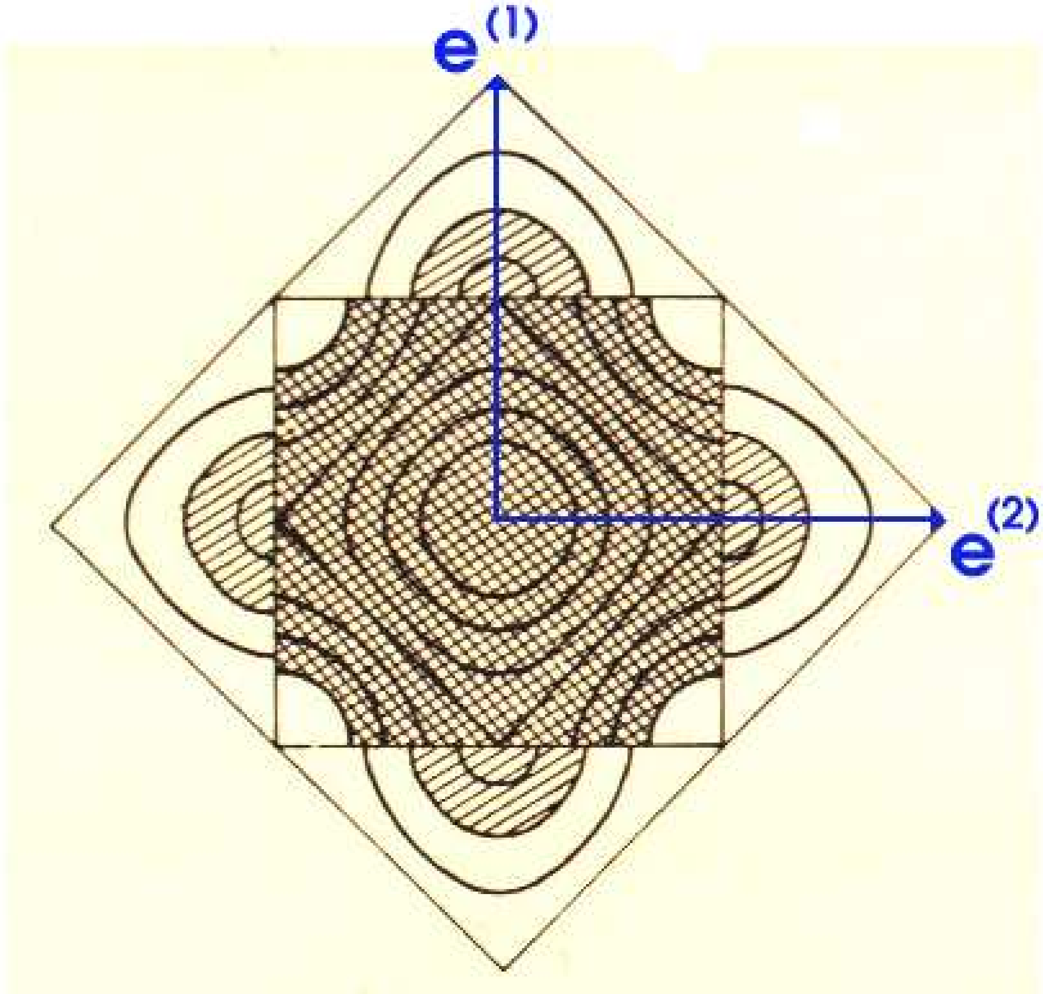

FIG. 3 – Fermi surface in momentum space

The corresponding annihilation and creation operators

$$\hat{c}_{\sigma k} = \sum_{\vec{r}} \varphi_k^*\{\vec{r}\}\hat{\psi}_\sigma\{\vec{r}\}\,, \quad \hat{c}^\dagger{}_{\sigma k} = \sum_{\vec{r}} \varphi_k\{\vec{r}\}\hat{\psi}^\dagger{}_\sigma\{\vec{r}\} \quad (23)$$

will be subject to the standard anticommutation relations

$$[\hat{c}_{\sigma k}, \hat{c}_{\sigma' k'}]_+ = 0\,, \qquad [\hat{c}^\dagger{}_{\sigma k}, \hat{c}^\dagger{}_{\sigma' k'}]_+ = 0\,, \qquad (24)$$

$$[\hat{c}^\dagger{}_{\sigma k}, \hat{c}_{\sigma' k'}]_+ = \delta_{\sigma\sigma'}\delta_{kk'}\,, \qquad (25)$$

In terms of these new operators the original position dependent annihilation and creation operators will be given by

$$\hat{\psi}_\sigma\{\vec{r}\} = \sum_k \varphi_k\{\vec{r}\}\hat{c}_{\sigma k}\,, \quad \hat{\psi}^\dagger{}_\sigma\{\vec{r}\} = \sum_k \varphi_k^*\{\vec{r}\}\hat{c}^\dagger{}_{\sigma k}. \quad (26)$$

It is to be noted that the discrete summations used here correspond in the continuum limit to volume integrals in ordinary space and wavenumber (pseudo - momentum) space respectively. On the understanding that the mesoscopic sample box in ordinary space has unit volume (with respect to an appropriately chosen measuring system such as c.g.s.) and that the dimension of the corresponding reciprocal box in wavenumber space is specified by a momentum cut off, k_c say, the correspondence will be given by

$$\sum_{\vec{r}} \leftrightarrow \left(\frac{k_c}{\pi}\right)^3 \int \mathrm{d}^3 r\,, \qquad \sum_{k} \leftrightarrow \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3 k\,. \quad (27)$$

A finite value for the regularisation scale k_c – which corresponds to a smoothing lengthscale $\lambda_c = 2\pi/k_c$ – is needed in for the avoidance of ultra-violet divergences, and will be provided by physical considerations.

The smoothing lengthscale for the neutrons may be comparable with their Compton wavelength. The orthonormality conditions (20) imply that the spin dependent number density operator $\hat{n}_\sigma\{\vec{r}\}$ will have space integral

$$\hat{\bar{n}}_\sigma = \sum_{\vec{r}} \hat{n}_\sigma\{\vec{r}\} = \sum_{k} \hat{c}^{\dagger}{}_{\sigma k}\hat{c}_{\sigma k} \,. \tag{28}$$

In terms of the independent particle contibution

$$\hat{\mathcal{H}}_{\rm ind}\{\vec{r}\} = \hat{\mathcal{H}}_{\rm kin}\{\vec{r}\} + \hat{\mathcal{H}}_{\rm pot}\{\vec{r}\}\,, \tag{29}$$

the total energy operator, in the absence of the interaction contribution, will be expressible in standard form as

$$\sum_{\vec{r}} \hat{\mathcal{H}}_{\rm ind}\{\vec{r}\} = \sum_{k} \mathcal{E}_k\, \hat{c}^{\dagger}{}_{\sigma k}\hat{c}_{\sigma k} \,. \tag{30}$$

To get the mean current (over the unit volume sample under consideration) as given by the operator (15) , we take the expectation value

$$\langle|\hat{\bar{n}}^i_\sigma|\rangle = \sum_{\vec{r}} \langle|\hat{n}^i_\sigma\{\vec{r}\}|\rangle \,, \tag{31}$$

for a state $|\,\rangle$ satisfying the simplicity condition that except for the diagonal contributions for which $\sigma' = \sigma$, $l_i = k_i$ the contributions of the expectation values $\langle|\hat{c}^\dagger{}_{\sigma k}\hat{c}_{\sigma' l}|\rangle$ will be negligible. We thus obtain an expression of the form

$$\langle|\hat{\bar{n}}^i_\sigma|\rangle = \sum_{k} \langle|\hat{n}_{\sigma k}|\rangle v^i_k \,, \tag{32}$$

in which v^i_k is the relevant pseudo-momentum dependent velocity average.

This velocity average will be given by

$$v_k^i = \sum_{\vec{r}} \frac{1}{2im\{\vec{r}\}} \gamma^{ij} \Big(\varphi_k^*\{\vec{r}\} \nabla_j \varphi_k\{\vec{r}\} - \varphi_k\{\vec{r}\} \nabla_j \varphi_k^*\{\vec{r}\} \Big) , \tag{33}$$

which is mathematically equivalent to the well known, albeit less intuitively obvious, group velocity formula that is given by (2) in terms of the single particle energy introduced in (21) as

$$v_k^i = \frac{\partial \mathcal{E}_k}{\partial k_i} . \tag{34}$$

4 Conducting reference state

The states in which we are interested are those that minimise the expected total energy $\langle|\hat{H}|\rangle$ subject not only to the usual constraint that there should be a fixed given value of the corresponding total expected particle number

$$\langle|\hat{n}|\rangle = \sum_{\sigma} \langle|\hat{n}_\sigma|\rangle\,, \tag{35}$$

but also, since we are concerned with non-static – conducting – configurations, to the reqirement that there should also be a fixed given value of the total

$$\langle|\hat{\tilde{n}}^{i}|\rangle = \sum_{\sigma} \langle|\hat{\tilde{n}}_{\sigma}^{\,i}|\rangle\,, \tag{36}$$

of the expected current with spin components defined by (32) .

Imposing these constraints via Laplace multiplyers μ and p_i , the problem becomes unconstrained minimisation of the combination

$$\langle|\hat{H}'|\rangle = \langle|\hat{H}|\rangle - \mu\langle|\hat{\overline{n}}|\rangle - p_i\langle|\hat{\widetilde{n}}^{\,i}|\rangle\,. \qquad (37)$$

In the absence of the pair coupling term, this reduces to the form

$$\langle|\hat{H}'_{\rm ind}|\rangle = \langle|\hat{H}_{\rm ind}|\rangle - \mu\langle|\hat{\overline{n}}|\rangle - p_i\langle|\hat{\widetilde{n}}^{\,i}|\rangle\,. \qquad (38)$$

in which by (30) that the effective Hamiltonian $\hat{H}'_{\rm ind}$ will have form

$$\hat{H}'_{\rm ind} = \sum_k \mathcal{E}'_k\,\hat{c}^{\dagger}{}_{\sigma k}\hat{c}_{\sigma k}\,. \qquad (39)$$

The effective energies will be given by

$$\mathcal{E}'_k = \mathcal{E}_k - \mu - p_i v_k{}^i \,, \tag{40}$$

which in the limit for which the squared momentum displacement $|p|^2 = \gamma^{ij} p_i p_j$ is small can be expressed as

$$\mathcal{E}'_k = \mathcal{E}_{k-p} - \mu + \mathcal{O}\{|p|^2\} \,. \tag{41}$$

The expectation value of the quantity given by (39) will evidently be minimised by choosing the state vector $|\rangle$ in such a way that the expectation $\langle|\hat{n}_{\sigma k}|\rangle = \langle|\hat{c}^\dagger{}_{\sigma k}\hat{c}_{\sigma k}|\rangle$ has its maximum value, namely 1, whenever $\mathcal{E}'_k$ is negative, and its minimum value, namely zero, whenever $\mathcal{E}'_k$ is positive.

It can thus be seen from (41) that the effect of the current will consist just of a uniform shift of the distribution in momentum space by an amount given by the infinitesimal momentum covector p_i . Such a state is characterised by the state conditions

$$\hat{c}^{\dagger}{}_{\sigma k}\hat{c}_{\sigma k}|\rangle = n_{\sigma k}|\rangle \,. \tag{42}$$

with the eigenvalues given as a Heaviside distribution by

$$n_{\sigma k} = \vartheta\{-\mathcal{E}'_k\} \,. \tag{43}$$

This state satisfies the condition for aplication of the formula (32) , having a well defined mean current value given (in accordance with our previous evaluation [1] in a first quantised framework) by

$$\overline{n}^i_\sigma = \sum_k v^i_k \, \vartheta\{-\mathcal{E}'_k\} \,. \tag{44}$$

When the current is small, it can be seen from (2) and (40) that this will be expressible to first order in terms of the static limit value

$$n_{\sigma k} = \vartheta\{\mu - \mathcal{E}_k\}\,. \tag{45}$$

of the distribution (43) as

$$\overline{n}^i_\sigma = p_j \sum_k \frac{\partial \mathcal{E}_k}{\partial k_i \partial k_j}\; n_{\sigma k} + \mathcal{O}\{|p|^2\}\,. \tag{46}$$

5 Bogoliubov pairing functions

Up to this point what we have just translated the work of our preceding article [1] from first quantised to the second quantised formalism needed for inclusion of for pairing interactions, which can be done in a BCS type model by taking the interaction contribution in (7) to have the form

$$\hat{\mathcal{H}}_{\rm int}\{\vec{r}\} = \Delta\{\vec{r}\}\hat{\psi}^{\dagger}{}_{\uparrow}\{\vec{r}\}\hat{\psi}^{\dagger}{}_{\downarrow}\{\vec{r}\} + \Delta^{*}\{\vec{r}\}\hat{\psi}_{\downarrow}\{\vec{r}\}\hat{\psi}_{\uparrow}\{\vec{r}\}\,, \tag{47}$$

where $\Delta\{\vec{r}\}$ is a position dependent complex potential that, in a "self consistent" model, should be given in the relevant reference state $|\rangle$ by an appropriate mean (with a Green function weighting of the same kind used for the "self consistent" evaluation of $V\{\vec{r}\}$) over neighbouring positions of the local crossed density expectation value $\langle|\hat{\psi}_{\downarrow}\{\vec{r}\}\hat{\psi}_{\uparrow}\{\vec{r}\}|\rangle$.

The mean complex phase of the function $\Delta\{\vec{r}\}$ is subject to an indeterminacy that can be resolved by fixing the phase in the specification of the wave operators. In a static configuration one would expect that this coupling potential $\Delta\{\vec{r}\}$ would share the ordinary periodicity property (10) of the lattice potential $V\{\vec{r}\}$, and moreover that the phase should be adjustable in such a way as to ensure that Δ becomes real.

Instead of using the representation (28) in terms of the simple Block wave functions $\varphi'_k\{\vec{r}\}$, in the approach introduced by Bogoliubov one seeks a more general representation whereby the single component Block waves are replaced by two component Bloch functions with components $\varphi_{k0}\{\vec{r}\}$ and $\varphi_{k1}\{\vec{r}\}$.

These are related to ordinary periodic functions $u_{k0}\{\vec{r}\}$, $u_{k1}\{\vec{r}\}$ by

$$\begin{pmatrix} \varphi_{k0}\{\vec{r}\} \\ \varphi_{k1}\{\vec{r}\} \end{pmatrix} = \mathrm{e}^{ik\cdot\vec{r}} \begin{pmatrix} u_{k0}\{\vec{r}\} \\ u_{k1}\{\vec{r}\} \end{pmatrix} . \tag{48}$$

The original representation (28) can thereby be replaced by a mixed representation involving a new set of position independent pseudo-particle annihilation and creation operators $\hat{\gamma}_{\sigma k}$ and $\hat{\gamma}^{\dagger}{}_{\sigma k}$ in terms of which we shall have

$$\hat{\psi}_{\uparrow}\{\vec{r}\} = \sum_{k} \left(\varphi_{k0}\{\vec{r}\}\hat{\gamma}_{\uparrow k} - \varphi^{*}_{k1}\{\vec{r}\}\hat{\gamma}_{\downarrow}{}^{\dagger}_{k} \right) , \tag{49}$$

$$\hat{\psi}_{\downarrow}\{\vec{r}\} = \sum_{k} \left(\varphi_{k0}\{\vec{r}\}\hat{\gamma}_{\downarrow k} + \varphi^{*}_{k1}\{\vec{r}\}\hat{\gamma}_{\uparrow}{}^{\dagger}_{k} \right) . \tag{50}$$

The Bogoliubov ansatz enable us to choose the new functions $\varphi_{k0}\{\vec{r}\}$ and $\varphi_{k1}\{\vec{r}\}$ in such a way as to simplify the expression

$$\hat{H}' = \sum_{\vec{r}} \hat{\mathcal{H}}'\{\vec{r}\}\,, \tag{51}$$

for the total effective Hamiltonian, in which

$$\hat{\mathcal{H}}'\{\vec{r}\} = \sum_{\sigma} \hat{\psi}^{\dagger}_{\sigma}\{\vec{r}\}\mathcal{H}'_{\text{ind}}\hat{\psi}_{\sigma}\{\vec{r}\} + \Delta\{\vec{r}\}\hat{\psi}^{\dagger}_{\uparrow}\{\vec{r}\}\hat{\psi}^{\dagger}_{\downarrow}\{\vec{r}\} + \Delta^{*}\{\vec{r}\}\hat{\psi}_{\downarrow}\{\vec{r}\}\hat{\psi}_{\uparrow}\{\vec{r}\}\,. \tag{52}$$

The independent particle contribution is given for a static configuration by

$$\mathcal{H}'_{\text{ind}} = \mathcal{H}_{\text{ind}} - \mu\,, \tag{53}$$

where, as before, μ is a Lagrange multiplier.

The purpose of th multiplier μ is to impose the constraint that the expectation of the total integrated number density should be held fixed when we apply the variation principle. It is to be remarked that in the presence of the BCS interaction term, the number operator $\hat{\bar{n}}$ will no longer exactly commute with the Hamiltonian, which implies that the state that minimisies the expectation of the effective Hamiltonian obtained in this way will not be an exact eigenstate either of the particle number or of the energy.

Substitution of (49) and (50) leads to an expansion that is expressible succinctly (dropping the explicit reference to dependence on the position vector $\vec{r}$) in the form

$$\hat{\mathcal{H}}' = \sum_k X_k + \sum_{kl} \left(Y_{kl} \sum_\sigma \hat{\gamma}^\dagger{}_{\sigma k} \hat{\gamma}_{\sigma k} + Z_{kl} \hat{\gamma}_{\uparrow k} \hat{\gamma}_{\downarrow l} - Z^* \hat{\gamma}_{\uparrow k}^\dagger \hat{\gamma}_{\downarrow l}^\dagger \right) . \tag{54}$$

The required (position dependent) coefficients $X_k\{\vec{r}\}$, $Y_{kl}\{\vec{r}\}$, $Z_{kl}\{\vec{r}\}$, will be given by

$$X_k = 2\varphi_{k1}\mathcal{H}'_{\text{ind}}\varphi^*{}_{k1} - \Delta\,\varphi_{k1}\varphi^*{}_{k0} - \Delta^*\varphi_{k0}\varphi^*{}_{k1}\,, \tag{55}$$

$$Y_{kl} = \varphi^*{}_{k0}\mathcal{H}'_{\text{ind}}\varphi_{l0} - \varphi^*{}_{l1}\mathcal{H}'_{\text{ind}}\varphi_{k1} + \Delta\,\varphi^*{}_{k0}\varphi_{l1} + \Delta^*\varphi^*{}_{k1}\varphi_{l0}\,, \tag{56}$$

$$Z_{kl} = \varphi_{k1}\mathcal{H}'_{\text{ind}}\varphi_{l0} + \varphi_{l1}\mathcal{H}'_{\text{ind}}\varphi_{k0} + \Delta\,\varphi_{k1}\varphi_{l1} - \Delta^*\varphi_{k0}\varphi_{l0}\,. \tag{57}$$

The essential step in the Bogoliubov procedure is to get rid of the contribution from the last set of coefficients.

The functions $\varphi_{k0}\{\vec{r}\}$, $\varphi_{k1}\{\vec{r}\}$ must therefore satisfy

$$\begin{pmatrix} \mathcal{H}'_{\rm ind} & \Delta \\ \Delta^* & -\mathcal{H}'^*_{\rm ind} \end{pmatrix} \begin{pmatrix} \varphi_{k0} \\ \varphi_{k1} \end{pmatrix} = \epsilon_k \begin{pmatrix} \varphi_{k0} \\ \varphi_{k1} \end{pmatrix} , \qquad (58)$$

where the eigenvalue ϵ_k is what will be seen to be interpretable as the relevant pseudoparticle energy. This system can be written in terms of the ordinarily periodic functions $u_{k0}\{\vec{r}\}$, and $u_{k1}\{\vec{r}\}$ introduced in (48) as

$$\begin{pmatrix} \mathcal{H}_k + V' & \Delta \\ \Delta^* & -\mathcal{H}^*_k - V' \end{pmatrix} \begin{pmatrix} u_{k0} \\ u_{k1} \end{pmatrix} = \epsilon_k \begin{pmatrix} u_{k0} \\ u_{k1} \end{pmatrix} , \qquad (59)$$

using the notation of (12) , where

$$V'\{\vec{r}\} = V\{\vec{r}\} - \mu \, . \qquad (60)$$

The foregoing specfication is incomplete, because the condition of satisfying (58) will evidently be preserved by interchanges of the form

$$\varphi^*{}_{k1} \leftrightarrow \varphi_{-k0}\,, \qquad \epsilon_k \leftrightarrow -\epsilon_k\,, \tag{61}$$

but this ambiguity is resolved by adoption of the usual postulate that the eigenvalues be positive,

$$\epsilon_k > 0\,. \tag{62}$$

The normalisation of the solutionsi is fixed in a manner that will automatically satisfy the integral relations expressible – restoring the explicit reference to the position dependence – as

$$\sum_{\vec r} \varphi_{k1}\{\vec r\}\varphi_{l0}\{\vec r\} = \sum_{\vec r} \varphi_{k0}\{\vec r\}\varphi_{l1}\{\vec r\}\,. \tag{63}$$

This is done by requiing that the new operators should satisfy anticommutation relations of the standard form

$$[\hat{\gamma}_{\sigma k}, \hat{\gamma}_{\sigma' k'}]_+ = 0 \, , \qquad [\hat{\gamma}^{\dagger}{}_{\sigma k}, \hat{\gamma}^{\dagger}{}_{\sigma' k'}]_+ = 0 \, , \tag{64}$$

$$[\hat{\gamma}^{\dagger}{}_{\sigma k}, \hat{\gamma}_{\sigma' k'}]_+ = \delta_{\sigma\sigma'}\delta_{kk'} \, , \tag{65}$$

which is achieved by fixing the amplitude of the (automatically mutually orthogonal) solutions so as to obtain

$$\sum_{\vec{r}} (\varphi^*{}_{k0}\{\vec{r}\}\varphi_{l0}\{\vec{r}\} + \varphi^*{}_{k1}\{\vec{r}\}\varphi_{l1}\{\vec{r}\}) = \delta_{kl} \, . \tag{66}$$

The foregoing ansatz reduces the effective Hamiltonian simply to

$$\hat{H}' = \sum_{\sigma,k} \epsilon_k \left(\hat{\gamma}^{\dagger}{}_{\sigma k} \hat{\gamma}_{\sigma k} - \sin^2\theta_k \right) = \sum_{\sigma,k} \epsilon_k \left(\cos^2\theta_k - \hat{\gamma}_{\sigma k} \hat{\gamma}^{\dagger}{}_{\sigma k} \right) \tag{67}$$

in which the Bogoliubov angle, θ_k, is given, for each value of k_i, by

$$\cos^2\theta_k = \sum_{\vec{r}} \varphi^*{}_{k0}\{\vec{r}\}\varphi_{k0}\{\vec{r}\}\,, \quad \sin^2\theta_k = \sum_{\vec{r}} \varphi^*{}_{k1}\{\vec{r}\}\varphi_{k1}\{\vec{r}\}\,. \tag{68}$$

Minimising expectation of (67) gives the pseudo-vacuum reference state $|\,\rangle$, which contains none of the pseudoparticles created by the operators $\hat{\gamma}^{\dagger}{}_{\sigma k}$:

$$\hat{\gamma}_{\sigma k}|\,\rangle = 0\,. \tag{69}$$

6 The BCS ansatz

Since (particularly for the middle layers of a neutron star crust, where the effective mass enhancement is likely [1] to be most important) we are still far from having a sufficiently knowledge of the solutions $\varphi_k\{\vec{r}\}$ for the independent particle model, it will evidently take some time before we can hope to obtain a complete evaluation of the solutions for the coupled equations for $\varphi_{k0}\{\vec{r}\}$ and $\varphi_{k1}\{\vec{r}\}$ using an accurate estimate of the coupling coefficient $\Delta\{\vec{r}\}$. In the meanwhile, as an immediately available approximation, offering the best that can be hoped for as a provisional estimate in the short run, we can use an ansatz of the standard BCS kind.

The standard BCS ansatz is a prescription of the form

$$\varphi_{k0}\{\vec{r}\} = \varphi_k\{\vec{r}\}\cos\theta_k\ , \qquad \varphi_{k1}\{\vec{r}\} = \varphi_k\{\vec{r}\}\sin\theta_k\ , \tag{70}$$

where the single component wave functions $\varphi_k\{\vec{r}\}$ are the independent particle eigenfunctions. These can be seen to be solutions of the simple Schroedinger type equation

$$\mathcal{H}'_{\rm ind}\varphi_k = \mathcal{E}_k{}'\varphi_k\ , \tag{71}$$

for which, in the static case under consideration at this stage,

$$\mathcal{E}_k{}' = \mathcal{E}_k - \mu \tag{72}$$

where $\mathcal{E}_k$ is the ordinary Bloch energy value as introduced in (20) .

It can be seen that the ansatz (70) will provide an exact solution in the limit for which the relevant coupling field matrix elements

$$\Delta_{kl} = \sum_{\vec{r}} \varphi_k^*\{\vec{r}\}\Delta_{\vec{r}}\varphi_l\{\vec{r}\}\,, \tag{73}$$

reduce to diagonal form, so that we have

$$\Delta_{kl} = \Delta_k \delta_{kl}\,, \tag{74}$$

using the notation

$$\Delta_k = \sum_{\vec{r}} \varphi_k^*\{\vec{r}\}\Delta\{\vec{r}\}\varphi_l\{\vec{r}\}\,. \tag{75}$$

This will only be an approximation when $\Delta\{\vec{r}\}$ is a field of the radially dependent form that has been obtained [3] using a Wigner Seitz type model.

The relation (75) will hold exactly when the coupling constant has a uniformly constant value Δ (which by choosing the relevant phase can be taken without loss of generality to be real and positive) so that we shall simply have $\Delta\{\vec{r}\} = \Delta = \Delta_k$, Subject to the validity of (74) , the BCS ansatz (70) will reduce the Bogoliubov system of differential equations (58) to the purely algebraic eigenvalue system having the form

$$\begin{pmatrix} \mathcal{E}'_k & \Delta_k \\ \Delta_k & -\mathcal{E}'_k \end{pmatrix} \begin{pmatrix} \cos\theta_k \\ \sin\theta_k \end{pmatrix} = \epsilon_k \begin{pmatrix} \cos\theta_k \\ \sin\theta_k \end{pmatrix} . \qquad (76)$$

The required eigenvalue for this BCS equation is given by the well known formula

$$\epsilon_k = \sqrt{\mathcal{E}'_k{}^2 + \Delta_k{}^2} \, . \qquad (77)$$

The corresponding field amplitudes will be given by

$$\tan\theta_k = \frac{\epsilon_k - \mathcal{E}'_k}{\Delta_k}\,, \tag{78}$$

which implies

$$\cos^2\theta_k = \frac{\epsilon_k + \mathcal{E}'_k}{2\epsilon_k}\,, \qquad \sin^2\theta_k = \frac{\epsilon_k - \mathcal{E}'_k}{2\epsilon_k}\,. \tag{79}$$

The ansatz (70) rduces the Bogoliubov transformation to the simple form

$$\hat{c}_{\uparrow k} = \cos\theta_k\,\hat{\gamma}_{\uparrow k} - \sin\theta_k\,\hat{\gamma}^{\dagger}_{\downarrow -k}\,, \quad \hat{c}_{\downarrow_k} = \cos\theta_k\,\hat{\gamma}_{\downarrow k} + \sin\theta_k\,\hat{\gamma}^{\dagger}_{\uparrow -k}\,. \tag{80}$$

The Bogoliubov transformation is equivalenty expressible as

$$\hat{\gamma}_{\uparrow k} = \cos\theta_k\, \hat{c}_{\uparrow k} + \sin\theta_k\, \hat{c}_{\downarrow -k}^{\dagger}\,, \quad \hat{\gamma}_{\downarrow k} = \cos\theta_k\, \hat{c}_{\downarrow k} - \sin\theta_k\, \hat{c}_{\uparrow -k}^{\dagger}\,. \tag{81}$$

It follows that for the state $|\,\rangle$ characterised by (69) the expectation values of the wavenumber dependent number density operators $\hat{n}_{\sigma k}$ introduced in (28) will be given by

$$\langle|\hat{n}_{\sigma k}|\rangle = \sin^2\theta_k\,, \tag{82}$$

This result is interpretable as expressing the effect commonly described as a smearing of the Fermi surface, whereby the smoothed out momentum space distribution (82) replaces the hard cut off expressed by the Heaviside formula (45) that applies in limit when the BCS coupling coefficient is set to zero.

7 Formula for the mobility tensor

When the static contribution charcterised by (53) is extended by the inclusion of the current constraint term proportional to the momentum covector p_i in the effective energy (37) , it can be seen that – as in the independent particle limit – its effect to first order will still be given just by the corresponding uniform displacement in the space of Block wavenumbers k_i .

Whether or not the BCS coupling term is included, the first order effect of the current will be entirely taken into account just by the set of infinitesimal adjustments

$$\mathcal{E}'_k \mapsto \mathcal{E}'_{\{p\}k} = \mathcal{E}'_{k-p} \,. \tag{83}$$

The effect of the current will therefore be given to first order by the infinitesimal transformations

$$\theta_k \mapsto \theta_{\{p\}k} \,, \qquad \hat{\gamma}_{\sigma k} \mapsto \hat{\gamma}_{\{p\}\sigma k} \,, \qquad \big|\,\big\rangle \mapsto \big|_{\{p\}}\big\rangle \,, \tag{84}$$

The expectation value of the total current thus obtained in the current carrying perturbed state $\big|_{\{p\}}\big\rangle$ will be given by (32) as

$$\overline{n}^i_{\{p\}} = \sum_\sigma \langle_{\{p\}}|\hat{\overline{n}}^i_\sigma|_{\{p\}}\rangle = \sum_{\sigma,k} v_k{}^i \, \langle_{\{p\}}|\hat{n}^i_{\sigma k}|_{\{p\}}\rangle \,, \tag{85}$$

in which as the analogue of (82) we shall have

$$\langle_{\{p\}}|\hat{n}^i_{\sigma k}|_{\{p\}}\rangle = \sin^2\theta_{\{p\}k} \,. \tag{86}$$

Since there is no current in the unperturbed state $|\,\rangle$, we just get

$$\overline{n}^{i}_{\{p\}} = 2\sum_{k} v_{k}{}^{i}\, p_i \frac{\partial(\sin^2\theta_{\{p\}k})}{\partial p_i} + \mathcal{O}\{|p|^2\}\,. \qquad (87)$$

It follows from (83) that in the small $|p|$ limit we shall have

$$\frac{\partial \mathcal{E}'_{\{p\}k}}{\partial p_i} = -\frac{\partial \mathcal{E}'_{\{p\}k}}{\partial k_i} = -\frac{\partial \mathcal{E}_k}{\partial k_i} = -v_k{}^i\,, \qquad (88)$$

and hence that the partial derivative in (87) can be evaluated in this limit as

$$\frac{\partial(\sin^2\theta_{\{p\}k})}{\partial p_i} = -v^i_k \frac{\partial(\sin^2\theta_k)}{\partial \mathcal{E}'_k} = -v^i_k \frac{\partial(\sin^2\theta_k)}{\partial \mathcal{E}_k}\,, \qquad (89)$$

in which $\sin^2\theta_k$ is given as a function of the quantity $\mathcal{E}'_k = \mathcal{E} - \mu$ and of Δ_k by (79).

The conclusion to be drawn from this is that the value of the current will be given by an expression of the same form

$$\overline{n}^{i}_{\{p\}} = p_j \mathcal{K}^{ij} + \mathcal{O}\{|p|^2\} \,, \qquad (90)$$

as in (46) , in which the required mobility tensor will be given by the formula

$$\mathcal{K}^{ij} = -2 \sum_{k} \frac{\partial(\sin^2\theta_k)}{\partial \mathcal{E}_k} \, v_k{}^i v_k{}^j \ , \qquad (91)$$

in which, by (115) , the relevant coefficient will be given by

$$\frac{\partial(\sin^2\theta_k)}{\partial \mathcal{E}_k} = -\frac{\Delta^2}{2\epsilon_k{}^3} \, . \qquad (92)$$

The translation of the discrete summation formula (80) into the language of continuous integration is given by (4) .

Except at base of crust where there are exotic (e.g. "spagetti" or "lassagne" type) configurations, the mobility tensor will have isotropic form,

$$\mathcal{K}^{ij} = \mathcal{K}\gamma^{ij} \, , \tag{93}$$

in which, writing ${v_k}^2 = \gamma_{ij} {v_k}^i {v_k}^j$, we shall have

$$\mathcal{K} = \tfrac{1}{3}\gamma_{ij}\mathcal{K}^{ij} = -\frac{2}{3}\sum_k \frac{\partial(\sin^2\theta_k)}{\partial \mathcal{E}_k} {v_k}^2 \, . \tag{94}$$

For case of constant gap parameter, $\Delta_k = \Delta$, one gets

$$\mathcal{K}^{ij} = 2\sum_k \sin^2\theta_k \frac{\partial^2 \mathcal{E}_k}{\partial k_i \partial k_j} \, , \quad \mathcal{K} = \tfrac{2}{3}\sum_k \sin^2\theta_k \, \gamma_{ij} \frac{\partial^2 \mathcal{E}_k}{\partial k_i \partial k_j} . \tag{95}$$

This formula (95) for the case of uniform coupling is useful for the evaluation of the corresponding effective mass $m^\star$ as defined by

$$m^\star = n/\mathcal{K}\,, \tag{96}$$

in terms of the relevant total particle number density as given by the prescription

$$n = \sum_\sigma \langle|\hat{\bar{n}}_\sigma|\rangle = 2\sum_k \sin^2\theta_k\,, \tag{97}$$

in which, if we only wish to count unbound neutrons, the summation should be taken only for values above a lower cut off below which the states are bound so that the corresponding values of the velocity v_k will vanish. Note that the concept of an effective mass has often been a source of confusion as different definitions have been used in different contexts.

In terrestrial solid state physics main concern is electric charge (not mass) whose transport in electric field E_i is given by Ohm law

$$j^i = e\,n^i = \sigma^{ij}\mathrm{E}_j. \tag{98}$$

The conductivity tensor σ^{ij} relates macroscopic measurable quantities, but it depends on the dynamical evolution of the medium unlike the newly introduced mobility tensor $\mathcal{K}^{ij}$, on which the effective mass $m_\star$ is defined. Since the conductivity tensor is given by

$$\sigma^{ij} = e^2\tau\mathcal{K}^{ij} \tag{99}$$

where τ is a typical scattering time, it will diverge whenever the system is in a superconducting state while the mobility tensor remains well behaved.

The result (91) smears the Heaviside distribution in the formula obtained without pairing, but is otherwise similar, involving the same group velocity v^i_k given as the momentum space gradient of the energy distribution $\mathcal{E}_k$ by (2) not its analogue $\tilde{v}^i_k$ got by substituting ϵ_k in place of $\mathcal{E}_k$, namely

$$\tilde{v}^i_k = \frac{1}{\hbar}\frac{\partial \epsilon_k}{\partial k_i} \, . \qquad (100)$$

This latter "pseudovelocity" is a mean velocity between particles and holes, since ϵ_k is the energy of a quasiparticle which is a mixture of particles and holes. When (as in the simple B.C.S. case) the gap parameter is independent of the momentum, this modified velocity will be given by the expression $\tilde{v}^i_k = v^i_k \mathcal{E}'_k / \epsilon_k$, so $\tilde{v}^i_k$ will vanish at the Fermi surface characterised by $\mathcal{E}_k = \mu$, where the number of particles is equal to the number of holes.

In the free particle limit for which, in so far as the unbound neutrons are concerned, the effect of the ionic potential wells characterised by the function $V\{\vec{r}\}$ is small (either because the wells occupy only a small part of the volume, as will be the case just above the neutron drip transition, or because the wells are shallow, as will be the case near the base of the crust) we shall have

$$\frac{\partial^2 \mathcal{E}_k}{\partial k_i \partial k_j} = \frac{1}{m}\gamma_{ij}\,, \tag{101}$$

where m is the uniform mass scale appearing in the kinetic energy operator, which will be comparable with, but for precision somewhat less that, the ordinary neutron mass $m_{\rm n}$. It can be seen by comparing (95) and (97) that in this free approximately uniform limit the effective mass for the unbound neutrons will be given simply by $m^\star = m$, regardless of whatever the value of the gap parameter Δ may be.

8 The property of superconductivity

Although there is much discussion of what is called "superconductivity" in astrophysically relevant contexts (including such exotic varieties as colour superconductivity in quark condensates) very little attention has been given to the actual property of superconductivity in the technical sense, meaning the possibility of having a relatively moving current that is effectively stable, or in stricter terminology *metastable*, with respect to small perturbations – such as would normally give rise to a dissipative damping mechanism of a resistive or viscous kind.

To find any discussion of the essentially important question of superconductivity in this technical sense – and in particular to find some attempt to provide a theoretical estimate the critical current value beyond which the superconductivity property will break down – we had to go back to relatively ancient literature concerned with electrons in laboratory metals, in which the issue is dealt with [4] in a rather brief and summary manner using heuristic arguments inspired by Landau's classic treatment [5] of the analogous problem in the context of superfluid Helium 4.

In the literature concerned on pulsars it has been taken for granted that neutron currents of the kind considered here actually are superconductiing in the sense of being metastable with respect to relevant kinds of perturbation. The purpose of this section is to demonstrate that this supposition of metastability should indeed be valid for the small amplitude currents in question. The issue is that of the stability, for small but finite values of the momentum covector p_i , of the superconducting reference state $\big|\,\big\rangle = \big|_{\{\mu p\}}\big\rangle$ that is characterised by the minimisation of the combination (37) .

The conducting state $\big|_{\{\mu p\}}\big\rangle$ was derived by minimising the energy expectation $\big\langle\big|\hat{H}\big|\big\rangle$ subject to the condition that the particle number expectation $\big\langle\big|\hat{\bar{n}}\big|\big\rangle$ and the current expectation $\big\langle\big|\hat{\tilde{n}}^{\,i}\big|\big\rangle$ were held fixed. It is physically evident that the particle number expectation $\big\langle\big|\hat{\bar{n}}\big|\big\rangle$ really will be preserved under the conditions of chemical equilibrium that are envisaged in the relevant applications, but why should $\big\langle\big|\hat{\tilde{n}}^{\,i}\big|\big\rangle$ be preserved ?

In a "normal" state the current expectation $\langle|\hat{\tilde{n}}^i|\rangle$ would tend to be damped by various scattering process. The crucial question is whether $\big|_{\{\mu p\}}\big\rangle$ will still minimise $\langle|\hat{H}|\rangle$ with respect to small relevant perturbations – subject as before to particle number preservation – when the prior assumption of preservation of $\langle|\hat{\tilde{n}}^i|\rangle$ is abandonned. Subject to the particle number conservation condition $\delta\langle|\hat{\bar{n}}|\rangle = 0\,,$ this stability requirement is equivalent to the condition of minimisation of $\langle|\hat{H}'|\rangle$ meaning that any admissible perturbation must satisfy

$$\delta\langle|\hat{H}'|\rangle > 0\,, \tag{102}$$

using, according to (37) , the notation $\hat{H}' = \hat{H}'_{\{p\}} + p_i\,\hat{\tilde{n}}^i\,.$

According to the reasonning of the previous section, the relevant adjustment of (68) will give us

$$\hat{H}'_{\{p\}} = \sum_{\sigma,k} \epsilon_{\{p\}k} \left(\hat{\gamma}^{\dagger}{}_{\{p\}\sigma k} \hat{\gamma}_{\{p\}\sigma k} - \sin^2\theta_{\{p\}k} \right) \;, \qquad (103)$$

so that the specification (36) provides the variation formula

$$\delta\langle|\hat{H}'_{\{p\}}|\rangle = \sum_{\sigma,k} \left(\epsilon_{\{p\}k} \delta\langle|\hat{\gamma}^{\dagger}{}_{\{p\}\sigma k} \hat{\gamma}_{\{p\}\sigma k}|\rangle + p_i\, v^i_k \delta\langle|\hat{n}_{\sigma k}|\rangle \right) \;, \qquad (104)$$

in which, for the BCS case, it can be seen from (77) that we shall have

$$\epsilon_{\{p\}k} = \sqrt{\mathcal{E}'_{\{p\}k}{}^2 + \Delta^2_k}\,. \qquad (105)$$

In this BCS case, the action on the conducting state $\big|_{\{\mu,p\}}\big\rangle$ of a typical quasiparticle creation operator $\hat\gamma^\dagger{}_{\{p\}\uparrow k}$ will provide only three non vanishing terms in the sum (104) , namely those given by

$$\delta\langle|\hat\gamma^\dagger{}_{\{p\}\uparrow k}\hat\gamma_{\{p\}\uparrow k}|\rangle = 1\,, \tag{106}$$

together with the number variation contributions

$$\delta\langle|\hat n_{\uparrow k}|\rangle = \cos^2\theta_{\{p\}k}\,, \qquad \delta\langle|\hat n_{\downarrow -k}|\rangle = -\sin^2\theta_{\{p\}k}\,. \tag{107}$$

The symmetry property $v^i_k = -v^i_{-k}$ ensures that the net energy contribution in (104) from such an excitation will be positive if and only if

$$\epsilon_{\{p\}k} + p_i\, v^i_k > 0\,. \tag{108}$$

It is to be noted that such an individual quasiparticle excitation may violate the requirement (8) that the number of real particles should be conserved, but it is evident from (107) that such violations may have either sign and so can be cancelled out by the combined effect of two or more elementary excitations. What, in a stable case, can not be cancelled out is the combined effect of several quasiparticle energy contributions in (104) : the quasiparticle energy contributions will always add up to give the positive result needed for stability provided the inequality (108) is satisfied for all admissible modes.

The stability condition (108) that we have derived in this way is the analogue, for a BCS type superconductor, of Landau's critical upper limit condition [5] for the relative flow rate of a simple superfluid. Squaring both sides of this inequality and substituting the expression (40) for $\mathcal{E}'_{\{p\}k}$ in (105) we see that in the BCS case there is a remarkable simplification (which does not seem to have been pointed out before) whereby the terms that are non linearly dependent on the momentum covector p_i cancel out, The stability condition will thereby take the form

$$2\, p_i\, v^i_k\, \mathcal{E}'_k < \epsilon^2_k\,. \qquad (109)$$

This can be rewritten in terms of the "pseudovelocity" introduced in (100) as

$$p_i\, \tilde{v}^i_k < \frac{1}{2}\epsilon_k\,, \qquad (110)$$

which is equivlent to

$$p_i \frac{\partial}{\partial k_i}\Big(\ln\{\epsilon^2_k\}\Big) < \hbar\,. \qquad (111)$$

For this to hold for all modes the magnitude p of the mean particle momentum covector p_i must satisfy an upper bound of the form

$$p < p_{\rm c}\,. \qquad (112)$$

For an approximately isotropic distribution depending only on the magnitude k of the wavenumber covector k_i, the critical value $p_{\rm c}$ will be given by

$$p_{\rm c} \approx \min_k \left\{ \frac{1}{v_k} \left(|\mathcal{E}'_k| + \frac{\Delta_k^2}{|\mathcal{E}'_k|} \right) \right\} , \qquad (113)$$

or equivalently by

$$\frac{\hbar}{p_{\rm c}} \approx \max_k \left\{ \frac{\partial}{\partial k} \left(\ln\{\epsilon_k^2\} \right) \right\} . \qquad (114)$$

Since $\mathcal{E}'_k$ vanishes on the Fermi surface, it is clear from (113) that $p_{\rm c}$ will also vanish – so that there will be no phenomenon of superconductivity – not only when the gap Δ_k vanishes everywhere, but even when it vanishes just in the neighbourhood of the Fermi surface.

When the gap value $\Delta_{\rm F}$ at the Fermi surface is non-zero but small compared with the other relevant energy scales – as will typically be the case – it can be seen that the minimum in (113) will be attained for energy values differing from the Fermi value by a small but finite positive amount that will be given approximately by $\mathcal{E}'_k \approx \pm\Delta_{\rm F}$. In such a case, it follows that the critical momentum value (113) will be expressible in terms of the value $v_{\rm F}$ of the group velocity magnitude v_k at the Fermi surface by the approximation

$$p_{\rm c} \approx 2\frac{\Delta_{\rm F}}{v_{\rm F}}\,. \qquad (115)$$

This more carefully derived formula is consistent with previous estimates based on vaguer heuristic arguments in the context of electon superconductivity in metals [4].

For a gap of the order of an Mev, in a region where the kinetic contribution to the Fermi energy has a typical value of the order of a few tens of Mev, the formula (115) implies a critical momentum value corresponding to a kinetic energy of relative motion of the order of hundreds of Kev per neutron. This is comparable with the total kinetic energy of rotation in the most rapidly rotating pulsars. However the relative rotation speeds of the neutron currents that are believed to be involved in pulsar glitch phenomena are very much smaller – by factors of 10^{-4} or even far less – than the absolute rotation speeds of the neutron star. In all such cases it may therefore be concluded that the superfluidity criterion (115) will be satisfied within an enormous confidence margin.

It is to be remarked that in the more thoroughly investigated context of laboratory superfluidity [5] Landau's simple linear formulation of the stability problem in terms just of phonons provides only an upper limit on the critical momentum whose true value is considerably reduced by the less mathematically tractable – since essentially non linear – effect of what are known as rotons. Analogous considerations presumably apply in the present context.

The upshot is that although our present treatment places the estimate (115) on a sounder footing than was provided by any previous work of which we are aware, it should still be considered just as an upper bound on the true critical value which is likely to be substantially reduced by non linear effects whose mathematical treatment is beyond the scope of the methods used here.

Despite this caveat, the prediction of genuine superconductivity in the context of glitching neutron star crusts should be considered to be very robust. The justification for such confidence is that – according to the considerations outlined in the preceding paragraph – the relevant magnitudes of the neutron currents in question correspond to values of the neutron momentum p that are extremely small compared with the order of magnitude given by (115). For such very low amplitude currents there is no obvious reason to doubt the validity of conclusions – including estimates of effective masses, as well as the prediction of genuine superfluidity – that are based on the simple kind of linearised treatment used here.

9 Conclusions

In the middle layers of the crust, where the inhomogeneities in $V\{\vec{r}\}$ will be important, our previous analysis neglecting the pairing gap lead to the prediction [1] of a strong "entrainment" effect whereby the value of $m^\star$ will become very large compared with m . We find here that this will not be significantly affected by the relevant gap Δ : so far as the effective mass is concerned neglect of superfluid pairing will be justifiable as a robust first approximation, at least for moderate values of the gap parameter.

The unimportance of pairing for entrainment is because, when Δ_k is small, the coefficient (92) will be very small except in a thin layer with width of the order of Δ_k near the Fermi surface locus where $\mathcal{E}_k = \mu$, so when the coupling is weak its effect will be entirely negligible. In sensitive cases for which the geometry of the energy contours near the Fermi surface is complicated by band effects, a moderately strong pairing effect might make a significant difference by smoothing out variations of the mobility tensor as a function of density, but it seem ulikely that this smearing effect would make much difference to the large scale averge properties of the mobility tensor.

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