Gravitational waves from eccentric binaries

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Some references

- T. Damour, A.Gopakumar, & B. R. Iyer, Phys. Rev. D **70**, 064028 (2004)
- R-M Memmesheimer, A.Gopakumar, & G. Schäfer, Phys. Rev. D **70, 104011 (2004)**
- C. Königsdörffer & A.Gopakumar, Phys. Rev. D **71**, 024039 (2005) ; Phys. Rev. D **73**, 124012 (2006)
- M. Tessmer & A. Gopakumar, *Accurate and efficient gravitational waveforms for certain galactic compact binaries*, to appear in MNRAS, (gr-qc/0610139)

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"Gravitationswellenastronomie"

Outline

- Astrophysical motivations

- How to construct GW templates for compact binaries in eccentric orbits & relevant inputs

- What is going on

Astrophysical Motivations

- Gravitational waves (GW) from inspiralling compact binaries (ICBs) should be observed by **LIGO, VIRGO** & proposed **LISA**
- ICBs are usually modeled as point particles in **quasi-circular** orbits & post-Newtonian approximation (PN) to GR accurately describe their dynamics
- For long lived & isolated compact binaries, the quasi-circular approximation is quite appropriate
- **However, there are several astrophysical scenarios that involve compact eccentric binaries**

Eccentric ICBs for LIGO/VIRGO ?

- Scenarios for Short Gamma ray Bursts

  Davies, Leavan & King (2005), Page et.al (2006)

  Grindlay, Zwart & McMillan (2006)

- Chaurasia & Bailes scenario (2005)

- Kozai Oscillations associated with hierarchical triplets

- Realistic dense star clusters simulations, employing N-body codes, required to create IMBHs.

  R. Spurzem & co-workers (work in progress)

Eccentric compact binaries for LISA

Several astrophysical scenarios indicate that LISA will hear GWs from Eccentric

- Galactic Stellar mass compact binaries
- Intermediate mass BH binaries
- Supermassive BH binaries

[See references in Königsdörffer & A.Gopakumar (2006)]

Therefore, it is desirable to construct GW templates for ICBs in eccentric orbits

GW templates for ICBs

- The widely employed gravitational wave templates are for detecting gravitational waves from non-spinning compact binaries in *quasi-circular orbits*.
- They consists of PN accurate expressions for h_+ & $h_\times$, supplemented by expressions giving *adiabatic* time evolution for the orbital phase and frequency $\phi(t)$ and $f(t)$
- $\phi(t)$ **&** $f(t)$ is known to 3.5PN order & h_+ & $h_\times$ to 2.5 PN order

  L. Blanchet, T. Damour, G. Esposito-Farèse, and B. R. Iyer, (2004) & (2005)

  K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah (2004)

  [See also references in these papers]

GW templates for eccentric CBs

- We are developing GW templates for compact binaries moving in **inspiralling eccentric orbits** that should be useful for GW data analysis community

- **The problem is more involved & non-trivial as we need to combine consistently *three* times scales**

- These are associated with the orbital motion, advance of the periastron & radiation reaction

  [we do not treat radiation-reaction in an adiabatic manner]

Damour, Gopakumar, Iyer, Königsdörffer, Memmesheimer, Schäfer, Tessmer, ...

Phasing: I

- **GW phasing: An accurate mathematical modeling of the continuous time evolution of the gravitational wave polarization states h_+ & $h_\times$**

- We employ **Damour's** mathematical formulation that resulted in the heavily employed **DD timing formula** for binary pulsars to do **GW phasing**

- **The dominant contributions to $h_\times$**
$$h_\times|_Q(r,\phi,\dot r,\dot\phi) =$$
$$-2\frac{G\,m\,\eta\,C}{c^4\,R'}\left\{\left(\frac{G\,m}{r}+r^2\,\dot\phi^2-\dot r^2\right)\sin 2\phi-2\dot r\,r\,\dot\phi\,\cos 2\phi\right\}$$ where
$$C=\cos i$$

- To construct GW templates, we require PN accurate $h_{+,\times}$ supplemented by explicit expressions describing the temporal evolution of the PN accurate relative motion, *i.e.* describing the explicit time dependences $r(t)$, $\phi(t)$, $\dot r(t)$, and $\dot\phi(t)$

Phasing: II

- The relative dynamics is given by $\ddot{\mathbf{r}} \equiv \mathcal{A} = \mathcal{A}_0 + \mathcal{A}'$

- $\mathcal{A}_0$ **is the 'conservative' (integrable) part &** $\mathcal{A}'$ **is the reactive perturbative part**

- The method first constructs a solution to the 'unperturbed' system, whose dynamics is governed by $\mathcal{A}_0$

- **A solution to the binary dynamics, governed by** $\mathcal{A}$**, is obtained by** ***varying the constants*** **in the generic solutions of the unperturbed system**

Phasing: III

- **For 3PN accurate dynamics, in the COM frame, there are 4 first integrals. The 3PN accurate energy and angular momentum of the binary (c_1 & c_2^i):**

$$
\begin{aligned}
c_1 &= \mathcal{E}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{v_1}, \mathbf{v_2})|_{\rm 3PN\ CM}\,, \\
c_2^i &= \mathcal{J}_i(\mathbf{x_1}, \mathbf{x_2}, \mathbf{v_1}, \mathbf{v_2})|_{\rm 3PN\ CM}\,,
\end{aligned}
$$

- The vectorial structure of c_2^i, indicates that the unperturbed motion takes place in a plane

- **This is true even when radiation reaction is present (at least to 3.5PN order)** & we can introduce polar coordinates in the plane of the orbit

Phasing: IV

- **The functional form for the solution to the unperturbed (3PN accurate) equations of motion**

$$r = S(l; c_1, c_2) \quad ; \quad \dot{r} = n \frac{\partial S}{\partial l}(l; c_1, c_2)\,,$$
$$\phi = \lambda + W(l; c_1, c_2) \quad ; \quad \dot{\phi} = (1+k)n + n \frac{\partial W}{\partial l}(l; c_1, c_2)\,,$$

The basic angles l and λ are given by

$$l = n(t - t_0) + c_l\,, \quad \lambda = (1+k)n(t - t_0) + c_\lambda$$

- l, λ & $S(l)$, $W(l)$, $\frac{\partial W}{\partial l}(l)$ are periodic in l with a period of $2\,\pi$
- The radial period $n = 2\,\pi/T_r$ & periastron advance parameter k are gauge invariant functions of c_1 & $c_2 = |c_2^i|$
- t_0 **is some initial instant and the constants** c_l **&** c_λ**, the corresponding values for** l **&** λ

Phasing: V

- **We construct the solution of the perturbed system, defined by $\mathcal{A}$, in the following way**

- We keep the same the functional form for r, $\dot{r}$, ϕ & $\dot{\phi}$, as functions of l & λ, but allow temporal variation in $c_1 = c_1(t)$ & $c_2 = c_2(t)$

- **Also, we have following definitions for l & λ**
  $l \equiv \int_{t_0}^{t} n\, dt \,+\, c_l(t) \qquad \lambda \equiv \int_{t_0}^{t} (1+k)\, n\, dt \,+\, c_\lambda(t)$

- **Note evolving quantities $c_l(t)$, & $c_\lambda(t)$**

- The four variables $\{c_1, c_2, c_l, c_\lambda\}$ replace the original four dynamical variables r, $\dot{r}$, ϕ & $\dot{\phi}$ and $\{c_\alpha\}$ satisfies first order evolution equations

Phasing: VI

- The explicit expressions for $\{dc_\alpha/dt\}$ read

$$\begin{aligned}
\frac{dc_1}{dt} &= \frac{\partial c_1(\mathbf{x},\mathbf{v})}{\partial v^i}\mathcal{A}'^i\,, \\
\frac{dc_2}{dt} &= \frac{\partial c_2(\mathbf{x},\mathbf{v})}{\partial v^j}\mathcal{A}'^j\,, \\
\frac{dc_l}{dt} &= -\left(\frac{\partial S}{\partial l}\right)^{-1}\left(\frac{\partial S}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial S}{\partial c_2}\frac{dc_2}{dt}\right), \\
\frac{dc_\lambda}{dt} &= -\frac{\partial W}{\partial l}\frac{dc_l}{dt}-\frac{\partial W}{\partial c_1}\frac{dc_1}{dt}-\frac{\partial W}{\partial c_2}\frac{dc_2}{dt}\,.
\end{aligned}$$

- The evolution of Eqs. for c_l & c_λ follow from the fact that we have same functional form for $\dot{r}$ & $\dot{\phi}$ in unperturbed & perturbed cases

Phasing: VII

- It is possible to split $c_\alpha(t) = \bar{c}_\alpha(t) + \tilde{c}_\alpha(t)$

  $\bar{c}_\alpha(t)$ represents a slow secular drift.

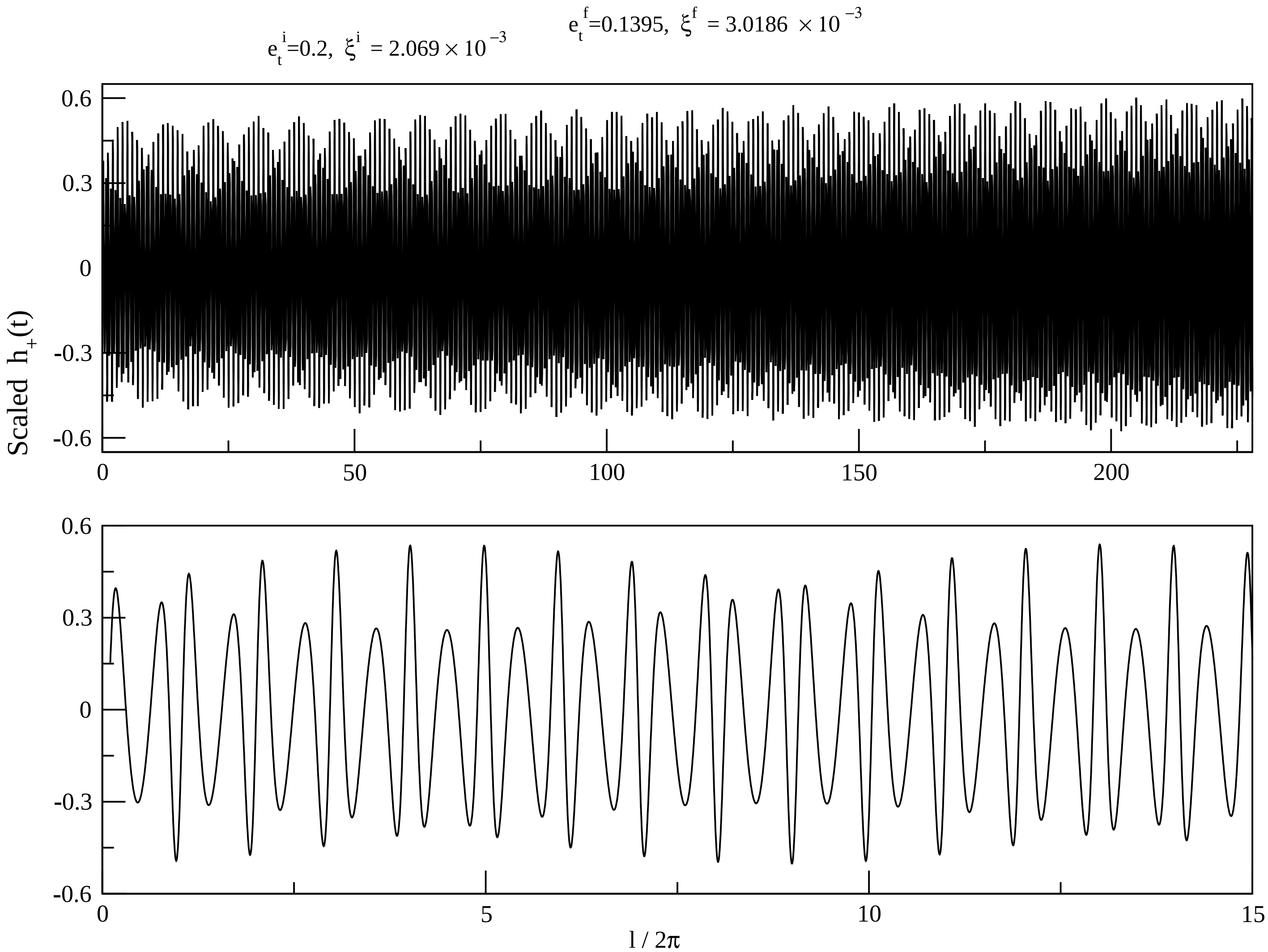
  $\tilde{c}_\alpha(t)$ represents periodic fast oscillations

- To explicitly perform GW phasing, we have solved evolution Eqs. for $\{\bar{c}_\alpha, \tilde{c}_\alpha\}$ to $\mathcal{O}(c^{-7})$ on the 3PN accurate description for the dynamical variables r, $\dot{r}$, ϕ & $\dot{\phi}$ entering the expressions for $h_\times$ & h_+ (Newtonian accurate in amplitude)

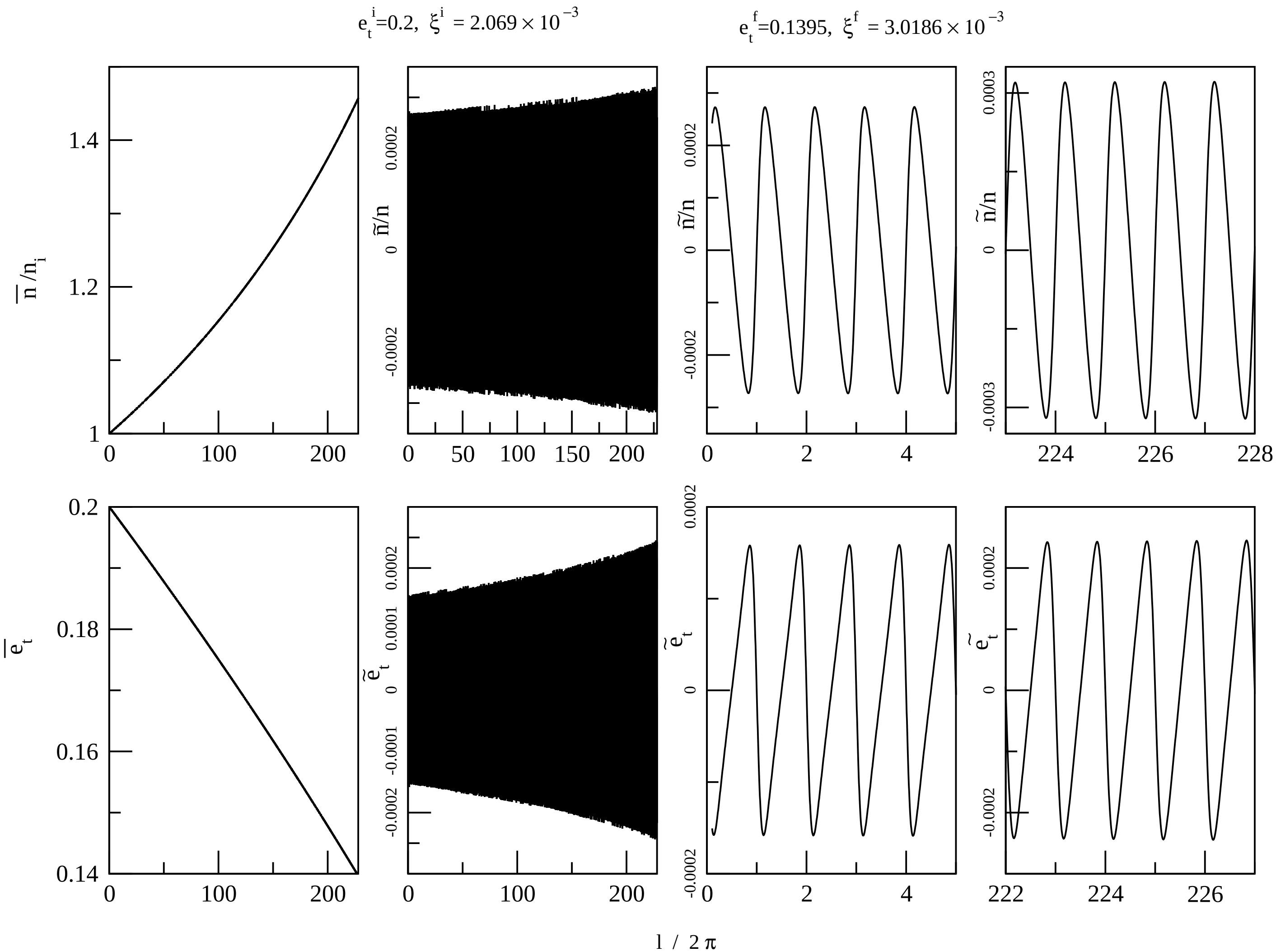
- **We have chosen to employ $\{n, e_t, c_l, c_\lambda\}$ as time dependent variables to describe the orbit**

  n **&** e_t**, associated with the 3PN accurate 'Kepler Eqn.' of generalized quasi-Keplerian representation, are expressible in** c_1 **&** c_2

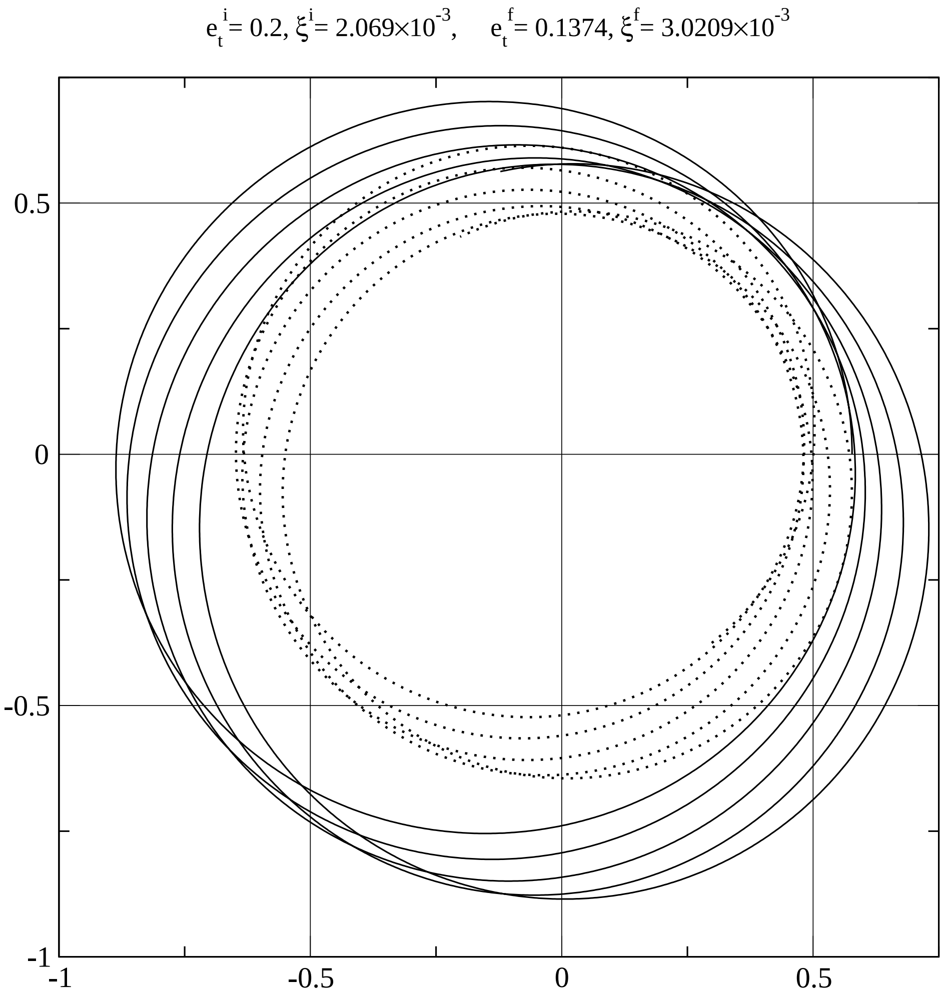
Scaled h_+ Vs # of orbital cycles for $\eta = 0.25$



$\{\bar{n}, \bar{e}_t, \tilde{n}, \tilde{e}_t\}$ Vs # of orbital cycles.



The orbital plot



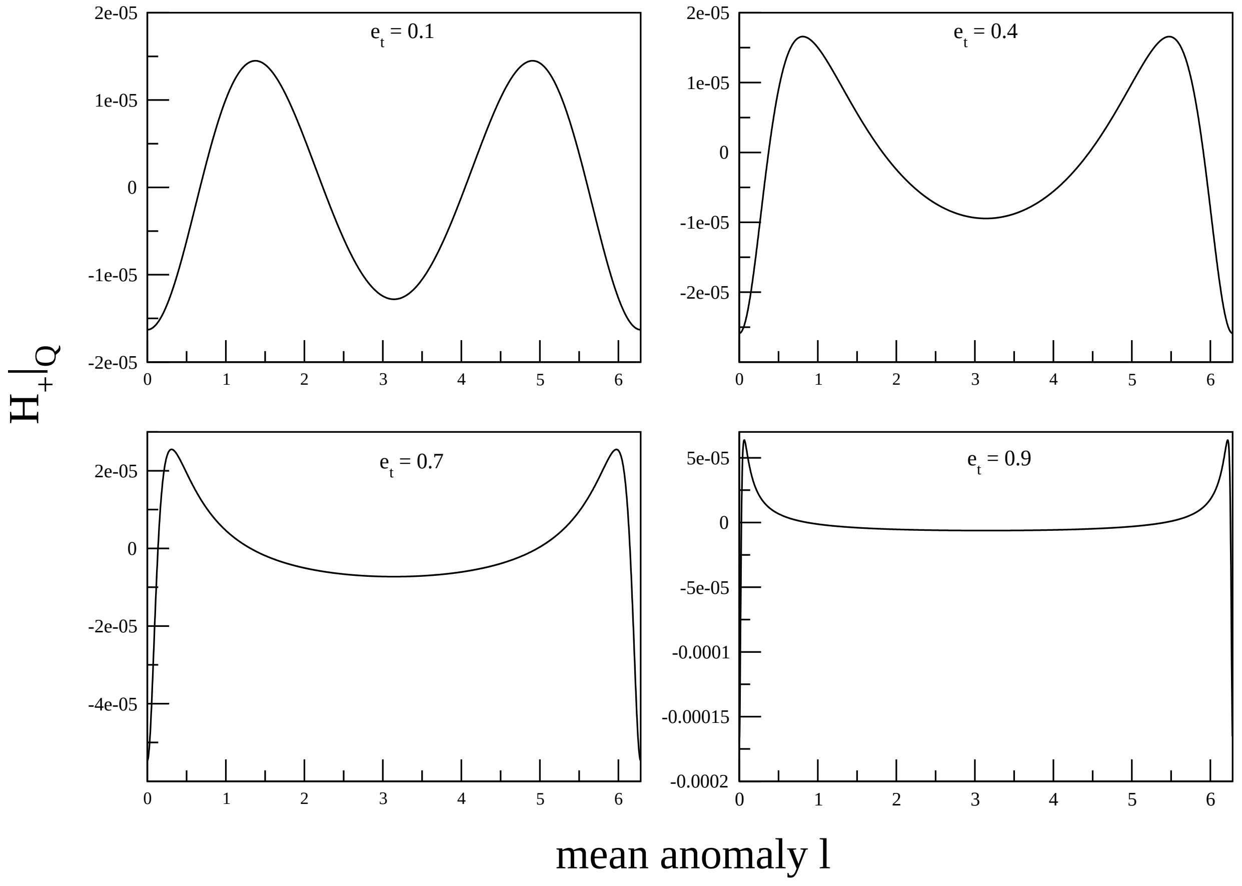
Details in T. Damour, A.Gopakumar, & and B. R. Iyer, Phys. Rev. D **70**, 064028 (2004) & C. Königsdörffer & A.Gopakumar, Phys. Rev. D **73**, 124012 (2006)

Ready-to-use LISA templates

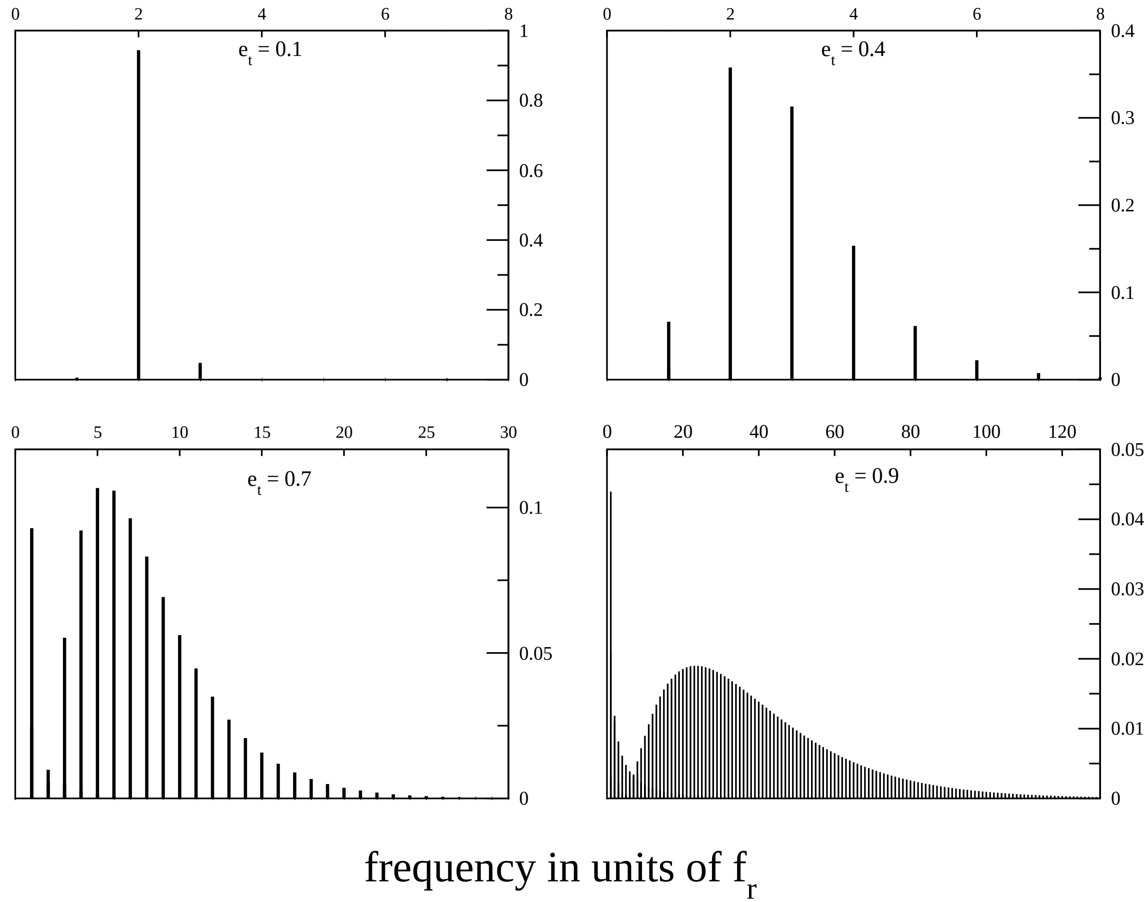
- Stellar-mass compact binaries in slowly precessing eccentric orbits are almost guaranteed sources of GWs for LISA
- We provided accurate and efficient ready-to-use GW templates for such galactic compact binaries that LISA will have to employ
- The crucial input is an accurate & efficient way of solving the 'Kepler Equation' $n(t - t_0) = u - e_t \sin u$ associated with PN accurate Keplerian-type parametric solution
- We employed numerical method of Mikkola that provides $u(t)$ with a relative error of $\sim 10^{-15}$ & valid for all values of t and e_t.

Details in M. Tessmer & A. Gopakumar (gr-qc/0610139)

Time evolution of scaled $h_+|_Q$ for various eccentricities:



The associated relative power spectrum to $h_+|_Q$:



- Recall that to do efficient phasing for ICBs in inspiralling eccentric orbits, we require certain semi-analytical solution to the conservative part of the orbital dyanamics: Keplerian type parametric solution

- GW phasing for eccentric ICBs is one of the important applications of PN accurate Keplerian type parametric solution

Let us take a look at the PN accurate Keplerian type parametric solution.

The Keplerian parametrization

- The KP gives parametric solution to Newtonian accurate orbital motion of a binary in an eccentric orbit

- Let R & ϕ be the components of the relative separation vector $\mathbf{R} = R(\cos\phi, \sin\phi, 0)$

$$
\begin{aligned}
R &= a(1 - e\cos u)\,, \\
\phi - \phi_0 &= v \equiv 2\arctan\left[\left(\frac{1+e}{1-e}\right)^{1/2} \tan\frac{u}{2}\right],
\end{aligned}
$$

- The explicit time dependence is provided by the Kepler equation

$$
l \equiv n(t - t_0) = u - e\,\sin u\,,
$$

The Keplerian parametrization

- The orbital elements a, e & $n = \frac{2\,\pi}{P}$ are expressible in terms of E, L, m_1 & m_2

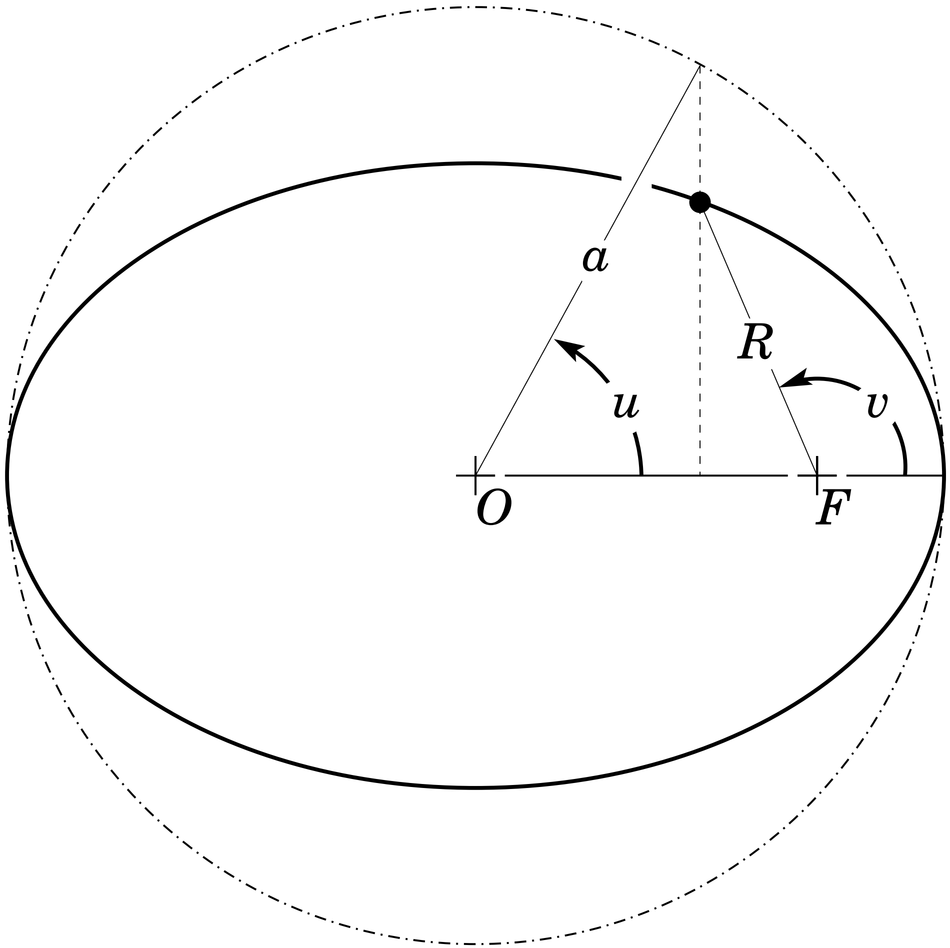
- u & v are the eccentric and true anomalies

- l is known as the mean anomaly

- **Several generations of western scientists worked on ‘Kepler equation’ roughly from 1650**

$$l \equiv n(t - t_0) = u - e\,\sin u\,,$$

The eccentric and true anomalies have geometrical interpretations



The quasi-Keplerian parametrization: I

- It is possible to find parametric solution to the conservative compact binary dynamics, given by $\mathcal{H}(\mathbf{r}, \hat{\mathbf{p}})|_{\rm 2PN}$

- The generalized quasi-Keplerian parametrization: **GQKP** Damour & Deruelle (1985), Damour & Schäfer (1988) and Schäfer & Wex (1993)

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- The generalized quasi-Keplerian parametrization: **GQKP**
Damour & Deruelle (1985), Damour & Schäfer (1988) and Schäfer & Wex (1993)

- **For the conservative 3PN accurate orbital dynamics of non-spinning compact binaries, $\mathcal{H}(\mathbf{r}, \hat{\mathbf{p}})|_{\rm 3PN}$ is available in ADM-type coordinates**
Damour, Jaranowski & Schäfer (2001)

- We derived 3PN accurate generalized quasi-Keplerian parametrization associated with $\mathcal{H}(\mathbf{r}, \hat{\mathbf{p}})|_{\rm 3PN}$
R.–M. Memmesheimer, A. Gopakumar, and G. Schäfer, Phys. Rev. D 70, 104011 (2004)

The 3PN accurate GQKP: I

- The orbital dynamics derivable from the 3PN accurate conservative Hamiltonian, in ADM–type coordinates in the center–of–mass frame, $\mathcal{H}(\mathbf{r}, \hat{\mathbf{p}})|_{\rm 3PN}$ allows 'Keplerian type' parametric solution:
- The 3PN accurate orbital energy & angular momentum $\mathbf{L}$, associated with $\mathcal{H}(\mathbf{r}, \hat{\mathbf{p}})|_{\rm 3PN}$, are conserved
- The conservation of $\mathbf{L} =>$ the motion is restricted to a plane, namely the orbital plane
- Introduce polar coordinates such that

$$\mathbf{r} = r(\cos\varphi, \sin\varphi)$$

The 3PN accurate GQKP: II

- The radial motion is parametrized by

$$r = a_r \left(1 - e_r \cos u\right)$$

- a_r, e_r 3PN accurate semi-major axis & 'radial eccentricity'
- These are expressible in terms of orbital energy E, angular momentum L and m_1 & m_2
- u is the eccentric anomaly

The 3PN accurate GQKP: III

- The angular motion is given by

$$
\begin{aligned}
\varphi - \varphi_0 &= (1+k)\,v + \left(\frac{f_{4\varphi}}{c^4} + \frac{f_{6\varphi}}{c^6}\right)\sin 2v + \left(\frac{g_{4\varphi}}{c^4} + \frac{g_{6\varphi}}{c^6}\right)\sin 3v \\
&\quad + \frac{i_{6\varphi}}{c^6}\sin 4v + \frac{h_{6\varphi}}{c^6}\sin 5v\,,
\end{aligned}
$$

where $v = 2\arctan\left[\left(\frac{1+e_\varphi}{1-e_\varphi}\right)^{1/2}\tan\frac{u}{2}\right]$ is the true anomaly

- k measures the advance of the periastron & e_φ is the 'angular eccentricity'
3PN accurate in E, L, m_1 & m_2.

- $f_{4\varphi}$, $f_{6\varphi}$, $g_{4\varphi}$, $g_{6\varphi}$, $i_{6\varphi}$, and $h_{6\varphi}$ are 2PN & 3PN order orbital functions expressible in terms of E, L, m_1 & m_2

The 3PN accurate GQKP: IV

- The 3PN accurate **'Kepler equation'**, which connects the eccentric anomaly to the coordinate time reads

$$
\begin{aligned}
l \equiv n\,(t - t_0) \quad &= \quad u - e_t \sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6} \right) (v - u) \\
&+ \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6} \right) \sin v + \frac{i_{6t}}{c^6} \sin 2v + \frac{h_{6t}}{c^6} \sin 3v
\end{aligned}
$$

- l is the mean anomaly, n the mean motion & e_t the 'time eccentricity'

- g_{4t}, g_{6t}, f_{4t}, f_{6t}, i_{6t} & h_{6t} are 2PN & 3PN order orbital functions expressible in terms of E, L, m_1 & m_2

The 3PN accurate GQKP: V

- When radiation reaction is included in the dynamics, orbital elements & functions evolve continuously with time

- The three eccentricities e_r, e_φ & e_t are connected by PN accurate relations involving E, L, m_1 and m_2

- The 3PN accurate GQKP for the conservative orbital motion of non-spinning compact binaries in eccentric orbits in both ADM-type & harmonic coordinates

The 3PN accurate GQKP: VI

- Explicit expression for a_r in ADM-type coordinates to 3PN

$$
\begin{aligned}
a_r &= \frac{1}{(-2\,E)} \Bigg\{ 1 + \frac{(-2\,E)}{4\,c^2}\,(-7+\eta) + \frac{(-2\,E)^2}{16 c^4} \Bigg[(1 + 10\,\eta + \eta^2) \\
&\quad + \frac{1}{(-2\,E\,h^2)} (-68 + 44\,\eta) \Bigg] + \frac{(-2\,E)^3}{192\,c^6} \Bigg[3 - 9\,\eta - 6\,\eta^2 \\
&\quad + 3\,\eta^3 + \frac{1}{(-2\,E\,h^2)} \left(864 + \left(-3\,\pi^2 - 2212\right)\eta + 432\,\eta^2 \right) \\
&\quad + \frac{1}{(-2\,E\,h^2)^2} \left(-6432 + \left(13488 - 240\,\pi^2\right)\eta - 768\,\eta^2 \right) \Bigg] \Bigg\} ,
\end{aligned}
$$

- The computation requires 3PN accurate expressions for dr/dt and $d\varphi/dt$ in terms of E, L & r, & they are polynomials of degree seven in $1/r$ at 3PN

Inclusion of spin-orbit effects

- **Königsdörffer & A. Gopakumar (2005) have included leading order relativistic spin-orbit interactions into the above parametrization**

- We have Keplerian type parametrization that describes not only the precessional motion of the orbit inside the orbital plane, but also the precessional motions of the orbital plane and the spins themselves

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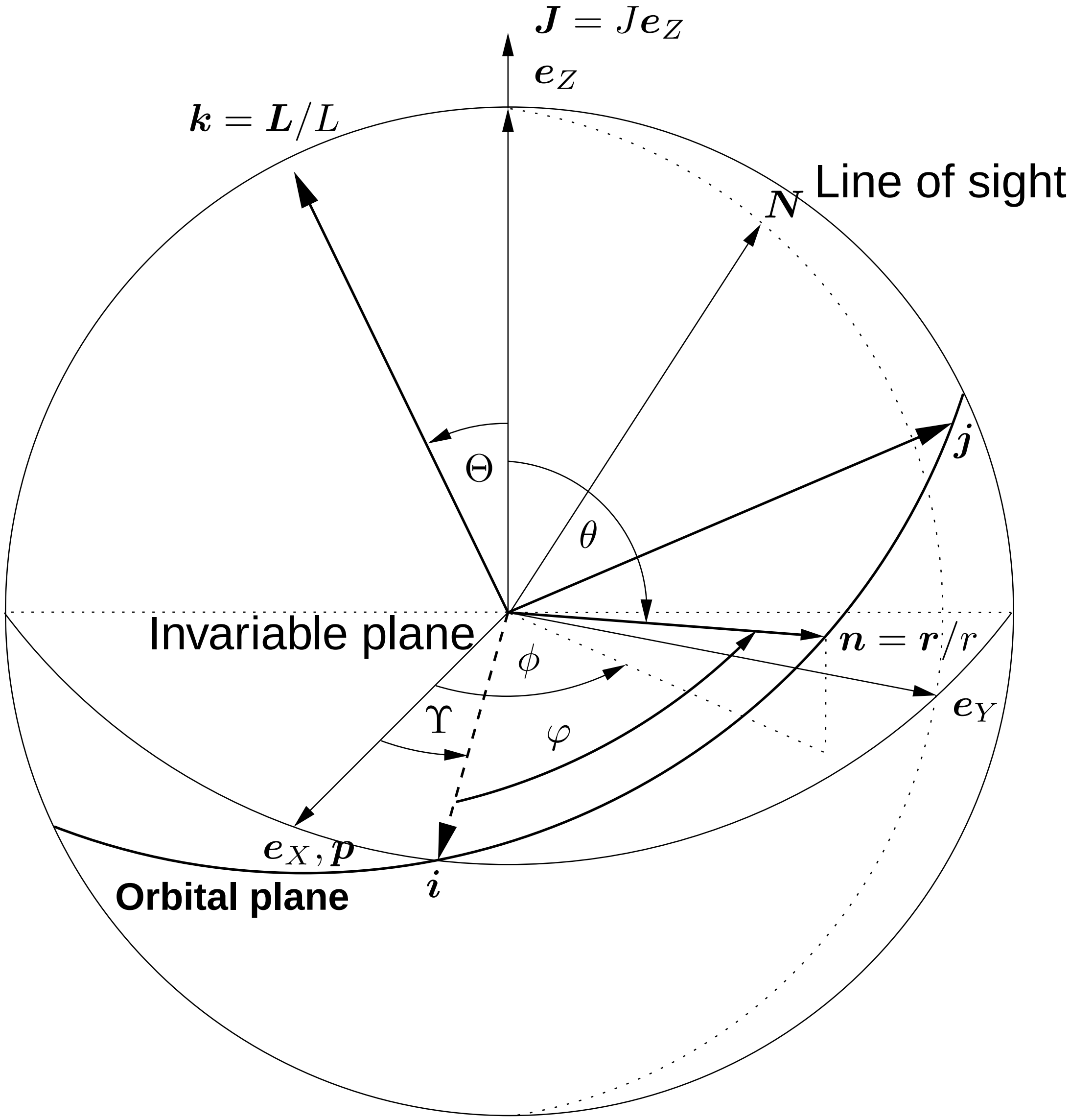
- We have Keplerian type parametrization that describes not only the precessional motion of the orbit inside the orbital plane, but also the precessional motions of the orbital plane and the spins themselves

- The orbital elements of the representation are explicitly given in terms of the conservedblue E, L, and a quantity that characterizes the leading order spin–orbit interactions

- A parametric solution to 3PN accurate orbital dynamics exists
  i) $m_1 \neq m_2$, $\boldsymbol{S}_1 \neq 0$ *or* $\boldsymbol{S}_2 \neq 0$ (Single spin case)
  ii) $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are arbitrary but $m_1 = m_2$

The Binary geometry



GQKP with spin-orbit coupling

- $$\begin{aligned} \boldsymbol{r}(t) &= r(t)\cos\varphi(t)\,\boldsymbol{i}(t) + r(t)\sin\varphi(t)\,\boldsymbol{j}(t)\,, \\ \boldsymbol{L}(t) &= L\boldsymbol{k}(t)\,, \\ \boldsymbol{S}(t) &= J\boldsymbol{e}_Z - L\boldsymbol{k}(t)\,, \end{aligned}$$

  The time evolution of the basic vectors $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ are given by

  $$\begin{aligned} \boldsymbol{i}(t) &= \cos\Upsilon(t)\boldsymbol{e}_X + \sin\Upsilon(t)\boldsymbol{e}_Y\,, \\ \boldsymbol{j}(t) &= -\cos\Theta\sin\Upsilon(t)\boldsymbol{e}_X + \cos\Theta\cos\Upsilon(t)\boldsymbol{e}_Y + \sin\Theta\,\boldsymbol{e}_Z\,, \\ \boldsymbol{k}(t) &= \sin\Theta\sin\Upsilon(t)\boldsymbol{e}_X - \sin\Theta\cos\Upsilon(t)\boldsymbol{e}_Y + \cos\Theta\,\boldsymbol{e}_Z \end{aligned}$$

  Θ, the precessional angle of $\boldsymbol{L}$, $= \frac{S\sin\alpha}{J}$

  $J = (L^2 + S^2 + 2LS\cos\alpha)^{1/2}$

  α is the angle between $\boldsymbol{L}$ & $\boldsymbol{S}$.

GQKP with spin-orbit coupling

- Time evolution for r, φ & Υ are given by

$$
\begin{aligned}
r &= a_r\,(1 - e_r \cos u)\ , \\
l \equiv n\,(t - t_0) &= u - e_t \sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)(v - u) \\
&\quad + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right)\sin v + \frac{i_{6t}}{c^6}\sin 2v + \frac{h_{6t}}{c^6}\sin 3v\,, \\
\varphi - \varphi_0 &= (1+k)v + \left(\frac{f_{4\varphi}}{c^4} + \frac{f_{6\varphi}}{c^6}\right)\sin 2v \\
&\quad + \left(\frac{g_{4\varphi}}{c^4} + \frac{g_{6\varphi}}{c^6}\right)\sin 3v + \frac{i_{6\varphi}}{c^6}\sin 4v + \frac{h_{6\varphi}}{c^6}\sin 5v\,, \\
\Upsilon - \Upsilon_0 &= \frac{\chi_{\mathrm{so}} J}{c^2 L^3}(v + e \sin v)
\end{aligned}
$$

The true anomaly $v = 2 \arctan\left[\left(\frac{1+e_\varphi}{1-e_\varphi}\right)^{1/2} \tan\frac{u}{2}\right]$.

The orbital elements & functions are expressible in terms of E, L, S, m_1, m_2 & α

Conclusions

We are trying to provide accurate and efficient GW templates for inspiralling eccentric binaries

Currently, we are

- investigating data analysis issues, relevant both for ground and space based GW detectors
- trying to include of spin-effects & to do phasing when orbital evolution is near the Last Stable Orbit
- working to model source scenarios