Self-force approach to EMRIs

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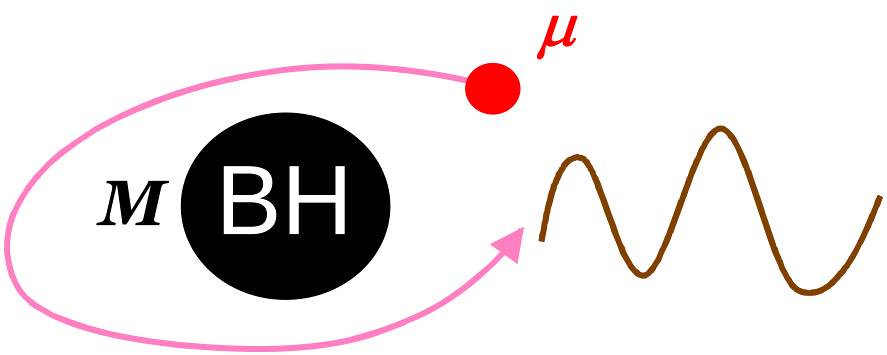
1. MiSaTaQuWa force for radiation reaction

- $$G^{\mu\nu}[g] = 8\pi G T^{\mu\nu}$$

  $$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \cdots$$

✧ $M >> \mu$

✧ v/c can be large



Energy-momentum of a point particle

$$T^{\mu\nu}(x) = \mu \int d\tau \, \dot{z}^\mu \dot{z}^\nu \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \qquad \left(\dot{z}^\mu = \frac{dz^\mu}{d\tau}\right)$$

Linear perturbation in μ

$$\delta G^{\mu\nu}\left[\boldsymbol{h}^{(1)}\right] = 8\pi G \boldsymbol{T}^{(1)\mu\nu}$$

$$\boldsymbol{T}^{(1)\mu\nu}(x) = \mu \int d\tau\, \dot{z}^{\mu}\dot{z}^{\nu} \frac{\delta^4(x - z(\tau))}{\sqrt{-g^{(0)}}} \qquad \left(\dot{z}^{\mu} = \frac{dz^{\mu}}{d\tau}\right)$$

geodesic on $g^{(0)}$

background metric

Master variable ζ:

$$\boldsymbol{\zeta} = \boldsymbol{h}^{(1)}_{\mu\nu} \quad \text{or} \quad {}_s\boldsymbol{\psi}^{(1)} \qquad ({}_s\boldsymbol{\psi} \sim \text{ a component of Weyl tensor})$$

$$\boldsymbol{\zeta} = \sum_{lm} \phi_{lm}(t, r) Y_{lm}(\Omega)$$

: expanded in spherical (spheroidal) harmonics

$$L[\boldsymbol{\zeta}] = S\left[\boldsymbol{T}^{(1)}\right]$$

Regge-Wheeler-Zerilli/Teukolsky eq.

From ζ, we can calculate:

- Waveform at infinity.
- $dE/dt|_{GW}$, $dL_z/dt|_{GW}$, etc. $\sim O\left((G\mu)^2\right)$

⇨ the orbit deviates from a geodesic on $g^{(0)}$

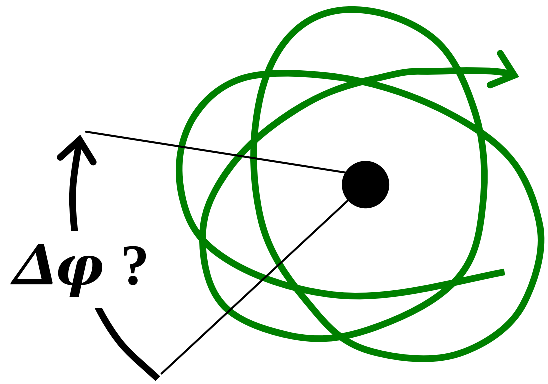
How can we incorporate this deviation?

- Use dE/dt & dL_z/dt to determine the evolution of the orbital parameters.

But, this cannot predict the phase shift in orbit

We need to evaluate dC/dt in the Kerr case.

C = Carter constant



- Evaluate self-force from $h_{\mu\nu}$ acting on the particle.

Self-force problem

For point particle,

$$\delta G^{\mu\nu}[\boldsymbol{h}] = 8\pi G \boldsymbol{T}^{\mu\nu} \quad \Longrightarrow \quad \boldsymbol{h}_{\mu\nu} \propto \frac{1}{|\boldsymbol{x} - \boldsymbol{z}(\boldsymbol{\tau})|}$$

$$\boldsymbol{h}_{\mu\nu}(x) \quad \textbf{diverges at} \quad \boldsymbol{x}^{\alpha} = \boldsymbol{z}^{\alpha}(\boldsymbol{\tau})$$

- self-force (back-reaction) in a curved background:

$$\mu \frac{D^2 z^\alpha}{d\tau^2} = F^\alpha[h] \approx \mu\, \delta\Gamma^\alpha_{\mu\nu}[h] \dot{z}^\mu \dot{z}^\nu = \mu \frac{1}{2}\left(h^\alpha{}_{\mu;\nu}(x) + ...\right) \dot{z}^\mu \dot{z}^\nu$$

~ geodesic eq. on $\boldsymbol{g}^{(0)} + \boldsymbol{h}$

singular !

● Breakdown of perturbation theory ?

Yes! & No!

- Yes, because a point particle is ill-defined in GR.

$\leftrightarrow$ Mass is non-renormalizable in GR

$$\lim_{r_0 \to 0} \left(m_{\mathrm{bare}} - \frac{G\, m_{\mathrm{bare}}^2}{r_0} \right)$$ has no well-defined limit.

- No, because $^{\exists}$ regular exact solution (BH) in GR.

$\leftrightarrow$ Mass renormalization is unnecessary

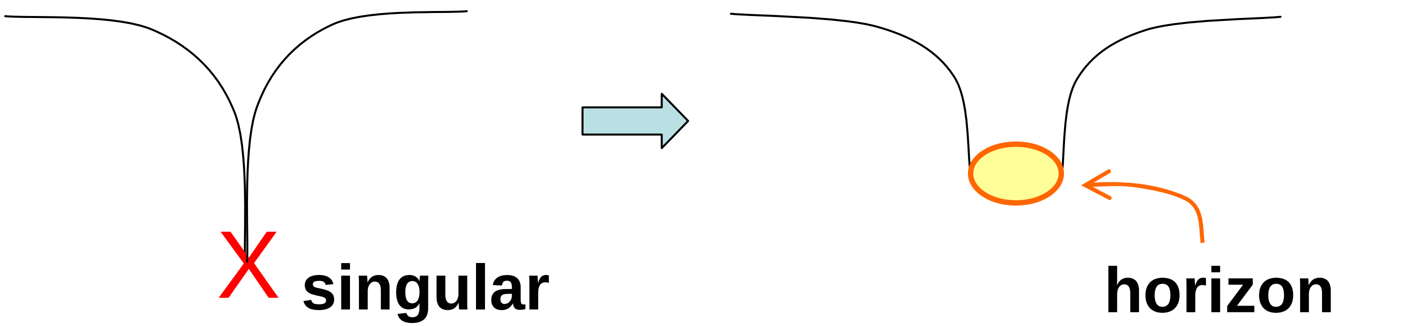
cf. EM theory:

point particle exists $\Longleftrightarrow$ mass is renormalizable

$$m_{\mathrm{phys}} = \lim_{r_0 \to 0} \left(m_{\mathrm{bare}} + \frac{e^2}{r_0} \right)$$: two parameters to tune the limit

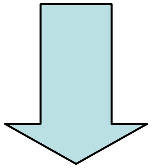
Namely, in GR:

- Identify the point particle with a BH solution of mass μ



- Embed the BH geometry in the linearly perturbed

metric $\boldsymbol{g}_{\mu\nu} = \boldsymbol{g}^{(0)}_{\mu\nu} + \boldsymbol{h}_{\mu\nu}$ **:** matching at $|x\text{-}z(\tau)| >> G\mu$



Matched Asymptotic Expansion

Thorne & Hartle ('84)

Simplest example

Consider a point particle in the flat background

background geodesic eq.

$$g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$$

$$h_{\mu\nu}(x) = \eta_{\mu\alpha}\eta_{\nu\beta}\frac{2G\mu\,(2\dot z^\alpha \dot z^\beta + \eta^{\alpha\beta})}{\dot z^0\,|\vec x - \vec z(\tau_{\mathrm{ret}})|}\,; \qquad \ddot z^\alpha(\tau) = 0$$

In the rest frame $\{X^a\}$ of the particle:

$$h_{ab}(X) = \eta_{ac}\eta_{bd}\frac{2G\mu\,(2\dot Z^c \dot Z^d + \eta^{cd})}{|\vec X|}\,; \quad \dot Z^a = (1,0,0,0)$$

This is just the Newtonian part of the Schwarzschild metric.

Thus a Schwarzschild BH of mass μ can be naturally matched to $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$ at $|X| >> G\mu$

⇨ EOM unchanged. No self-force correction to all orders in $G\mu$

In General Curved Background:

- Hadamard decomposition of $G_{(ret)}$ in harmonic (Lorenz) gauge

$$G^{\mu\nu}_{(\mathrm{ret})\alpha\beta}(x,z) = \theta(x^0 - z^0)\left[u^{\mu\nu}{}_{\alpha\beta}\ \delta(\sigma(x,z)) - v^{\mu\nu}{}_{\alpha\beta}\ \theta(-\sigma(x,z))\right]$$

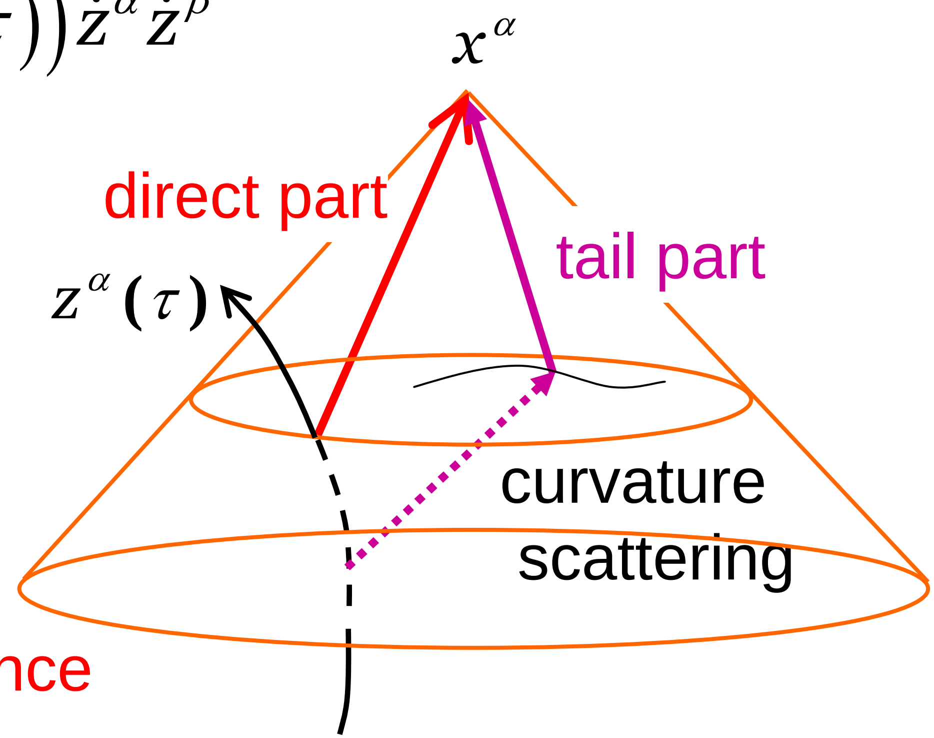
$\sigma(x,z)$: world interval between x and z $\left(\sim \frac{1}{2}(x\text{-}z)^2\right)$

$$h^{\mu\nu}_{(\mathrm{ret})}(x) = \mu\int d\tau\ G^{\mu\nu}_{(\mathrm{ret})\alpha\beta}(x,z(\tau))\dot{z}^\alpha \dot{z}^\beta$$

$u^{\mu\nu}{}_{\alpha\beta}$: direct part

$v^{\mu\nu}{}_{\alpha\beta}$: tail part

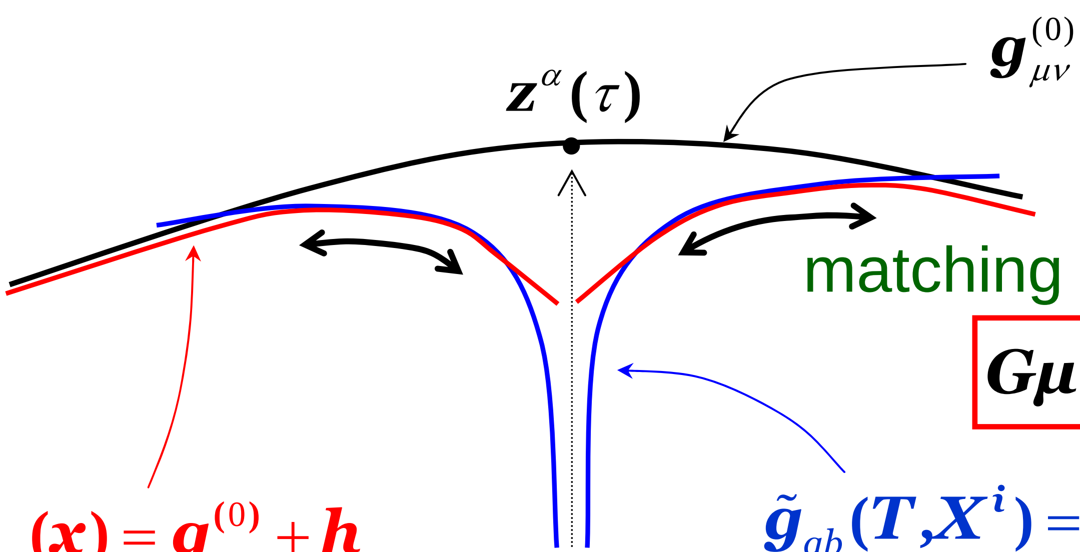
$$h^{\mu\nu}_{(\mathrm{ret})}(x) = h^{\mu\nu}_{(\mathrm{direct})} + h^{\mu\nu}_{(\mathrm{tail})}$$



$h^{\mu\nu}_{(\mathrm{direct})}$ contains divergence

$v^{\mu\nu}{}_{\alpha\beta}$ is a solution of source-free eq. but not $h^{\mu\nu}_{(\mathrm{tail})}$

Matched asymptotic expansion



$$\left(R^\alpha{}_{\beta\gamma\delta}[g^{(0)}] \sim \frac{1}{L^2} \right)$$

$$G\mu << X \sim \sqrt{G\mu\, L} << L$$

$$g_{\mu\nu}(x) = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

(external: valid at $|X| >> G\mu$)

$$\tilde{g}_{ab}(T, X^i) = H^{(\mathrm{BH})}_{ab} + \delta H_{ab}$$

(internal: valid at $|X| << L$)

- coordinate transformation: $g_{ab}(X) = \dfrac{\partial x^\mu}{\partial X^a} \dfrac{\partial x^\nu}{\partial X^b} g_{\mu\nu}(x)$

$$\sigma^{;\mu}(x, z(\tau)) \left(\approx -(x^\mu - z^\mu) \right) = -\left(f^\mu{}_i(T) X^i + f^\mu{}_{ij}(T) X^i X^j + \ldots \right)$$

$$\sigma^{;\mu}(x, z(\tau))\, \bar{g}_{\mu\alpha}(x, z) \dot{z}^\alpha = 0\ ;$$

$\bar{g}_{\mu\alpha}$: parallel transport bi-tensor

- identify g_{ab} with $\tilde{g}_{ab}$ in the matching region.

Regularized Gravitational Self-force

'MiSaTaQuWa' force: (named by Eric Poisson)

$$F^{\alpha}[h_{(\mathrm{tail})}(x)] \approx \frac{1}{2}(h^{\alpha}{}_{(\mathrm{tail})\,\mu;\nu}(x) + \ldots\ )\,\dot{z}^{\mu}\dot{z}^{\nu}$$

Mino, Sasaki and Tanaka ('97), Quinn and Wald ('99)

Tail part of the metric perturbation

$$h^{\mu\nu}_{(\mathbf{tail})}(x) \approx \int_{-\infty}^{\tau(x)} d\tau' \; v^{\mu\nu}{}_{\alpha\beta}\big(x, z(\tau')\big) T^{\alpha\beta}\big(z(\tau')\big)$$

Regularized self-force is determined by the tail part

E.O.M. with self-force = geodesic on $g^{\mu\nu} + h^{\mu\nu}_{(\mathrm{tail})}$

But $h^{\mu\nu}_{(\mathrm{tail})}(x)$ is NOT a solution of Einstein equations.

⇒ meaning of the metric $g^{\mu\nu} + h^{\mu\nu}_{(\mathrm{tail})}$ was unclear

Detweiler - Whiting's S-R decomposition

(improved over "direct-tail" decomposition) **PRD 67, 024025 (2003)**

$$G^{ret}(x,z) = 2\theta\left(x^0 - z^0\right) G^{sym}(x,z)$$

$$G^{sym}(x,z) = \frac{1}{8\pi}\left[u(x,z)\delta(\sigma) - v(x,z)\theta(-\sigma)\right]$$

$$G^{S}(x,z) = G^{sym}(x,z) + \frac{1}{8\pi}v(x,z) = \frac{1}{8\pi}\left[u(x,z)\delta(\sigma) + v(x,z)\theta(\sigma)\right]$$

$$h^{S}(x) = \int d^4x'\sqrt{-g}\; G^{S}(x,x')T(x')$$: satisfies pert eqs.

$$G^{R}(x,z) = G^{ret}(x,z) - G^{S}(x,z) = \left(G^{ret}(x,z) - G^{adv}(x,z)\right) - \frac{1}{8\pi}v(x,z)$$

$$h^{R}(x) = h^{ret}(x) - h^{S}(x)$$: satisfies source-free pert eqs.

$$\boxed{h^{R} - h^{\text{tail}} = O\left((x-z)^2\right)}$$ ⇒ Both give the same force

EOM = geodesic on $g^{(0)}_{\mu\nu} + h^{R}_{\mu\nu}$ → solution of (linearized) vacuum Einstein eqs.

2. Adiabatic approximation

Constants of motion for geodesics in Kerr

Kerr geometry:

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r}{\Sigma}\sin^2\theta\right)\sin^2\theta d\phi^2,$$

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2.$$

Two Killing vectors and One Killing tensor:

$$\xi^\mu_{(t)} = (1,0,0,0), \quad \xi^\mu_{(\phi)} = (0,0,0,1), \quad K_{\mu\nu} = 2\Sigma l_{(\mu} n_{\nu)} + r^2 g_{\mu\nu},$$

$$l^\mu = \left(\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \quad n^\mu = \frac{\Delta}{2\Sigma}\left(\frac{r^2+a^2}{\Delta}, -1, 0, \frac{a}{\Delta}\right),$$

$$m^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right).$$

Constants of motion:

$$E = -u^\alpha \xi^{(t)}_\alpha \qquad L = u^\alpha \xi^{(\phi)}_\alpha \qquad Q = K_{\alpha\beta} u^\alpha u^\beta \qquad C \equiv Q - (aE - L)^2$$

$$\dot{Q} = K_{\alpha\beta;\sigma} u^\alpha u^\beta u^\sigma = 0 \quad \because \quad K_{(\alpha\beta;\sigma)} = 0$$

← definition of Killing tensor

Radiation reaction to the Carter constant

Schwarzschild "constants of motion" $E, L_i \Leftrightarrow$ Killing vector

Conserved current for GW corresponding to Killing vector exists.

$$E_{GW} = \int d\Sigma^{\mu} t_{\mu\nu}^{(GW)} \xi^{\nu}$$

$$\dot{E} = -\dot{E}_{gw}$$ In total, conservation law holds.

Kerr conserved quantities $E, L_z \Leftrightarrow$ Killing vector

$Q \not\Leftrightarrow$ Killing vector

$$Q = C + (aE - L_z)^2 = K_{\mu\nu} u^{\mu} u^{\nu}$$

↑ Killing tensor

How can we evaluate dQ/dt?

Adiabatic approximation

$$T << \tau_{RR}$$

T: orbital period, τ_{RR} : radiation reaction timescale

- At lowest order, the trajectory is given by a geodesic specified by E, L_z, Q (Carter const.).

- We evaluate backreaction to E, L_z, Q by using radiative field rather than R field.

$$\left\langle \frac{D}{d\tau} X \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d\tau \frac{\partial X}{\partial u^{\alpha}} F^{\alpha}\left[h_{\mu\nu}^{(rad)} \right]$$

$$X = E, L_z, Q \qquad h_{\mu\nu}^{(rad)} = \frac{1}{2}\left[h_{\mu\nu}^{(ret)} - h_{\mu\nu}^{(adv)} \right]$$

$$G^{R,ret} \to \frac{1}{2}\left(G^{R,ret} - G^{R,adv} \right) = \frac{1}{2}\left(G^{ret} - G^{adv} \right) \equiv G^{rad}$$

·Radiative field does not have divergence at the location of the particle.

$$G^{rad} = \frac{1}{2}\left(G^{ret} - G^{adv}\right)$$

$$\boxed{\Box G^{adv} = \Box G^{ret} = \delta^4\left(x - x'\right)} \Rightarrow \boxed{\Box G^{rad} = 0}$$

- For *E* and L_z the results are consistent with the balance argument. (shown by Gal'tsov '82)

- For *Q*, it was proved that the use of the radiative field gives the correct long time average. (shown by Mino '03)

  - Key: under a transformation

$$\boxed{\left(t, r, \theta, \phi\right) \to \left(c_t - t, r, \theta, c_\phi - \phi\right)} \qquad c_t,\ c_\phi \ \cdots \ \text{constants}$$

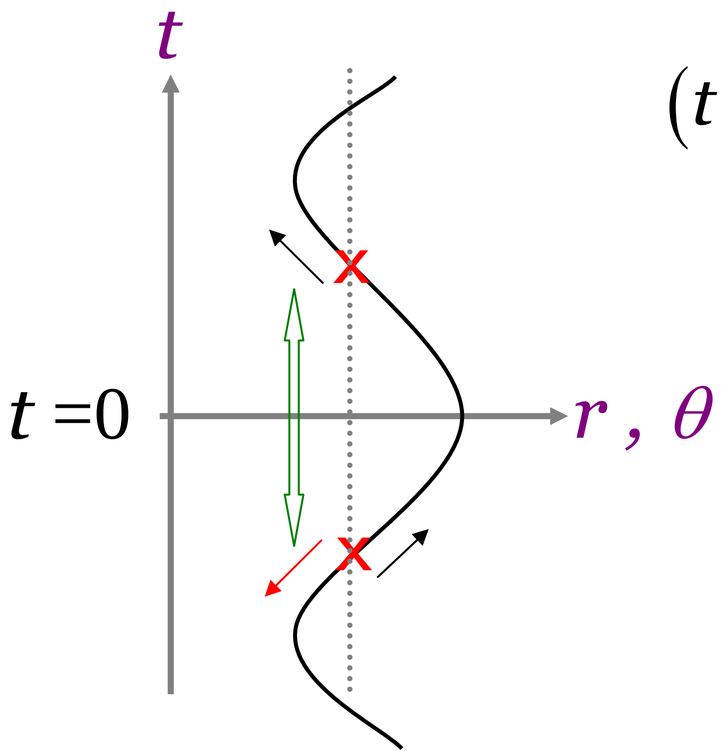
every geodesic is transformed into itself

GPS (Geodesic Preserving Symmetry) transformation

Mino, PRD67:084027 (2003)

For a given bounded geodesic, one can find a point at which $dr/dt = d\theta/dt = 0$, unless the r and θ periods are commensurable.

Set $t = \phi = 0$ at this point.



GPS transformation:

$$(t, r, \theta, \phi) \to (t', r', \theta', \phi') = (-t, r, \theta, -\phi)$$

⇩

$$G^{ret}(x', z') = G^{adv}(x, z)$$

$$G^{adv}(x', z') = G^{ret}(x, z)$$

In terms of matrix notation,

$$d\boldsymbol{x}(t) = -\boldsymbol{J}\, d\boldsymbol{x}(-t), \quad \boldsymbol{u}(t) = \boldsymbol{J}\,\boldsymbol{u}(-t),$$

$$\boldsymbol{h}^{(adv)}(t) = \boldsymbol{J}\,\boldsymbol{h}^{(ret)}(-t)\boldsymbol{J}$$

$$\boldsymbol{J} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} t & r & \theta & \phi \end{matrix}$$

⇓

$$\boldsymbol{F}^{(adv)}(t) = -\boldsymbol{J}\,\boldsymbol{F}^{(ret)}(-t)$$

⇓

$$\dot{Q}^{(adv)}(t) = 2\boldsymbol{u}\bullet\boldsymbol{K}\,\boldsymbol{F}^{(adv)}(t) = -2\boldsymbol{u}\bullet\boldsymbol{J}\,\boldsymbol{K}\,\boldsymbol{J}\,\boldsymbol{F}^{(ret)}(-t)$$

$$= -2\boldsymbol{u}\bullet\boldsymbol{K}\,\boldsymbol{F}^{(ret)}(-t) = -\dot{Q}^{(ret)}(-t)$$

$$\left(\boldsymbol{K} = \boldsymbol{J}\,\boldsymbol{K}\,\boldsymbol{J}\right)$$

⇓

$$\left\langle \dot{Q}^{(adv)}(t) \right\rangle = -\left\langle \dot{Q}^{(ret)}(t) \right\rangle$$

⇓

$$\left\langle \dot{Q}^{(ret)}(t) \right\rangle = \frac{1}{2}\left(\left\langle \dot{Q}^{(ret)}(t) \right\rangle - \left\langle \dot{Q}^{(adv)}(t) \right\rangle\right) = \left\langle \dot{Q}^{(rad)}(t) \right\rangle$$

Mino ‘03

dQ/dt formula

Sago, Tanaka, Hikida & Nakano ('05)

$$\frac{dQ}{d\tau} = 2u^{\mu} K_{\mu\nu} f^{\nu} \qquad f^{\alpha} = -\frac{1}{2}\left(g^{\alpha\beta} + u^{\alpha}u^{\beta}\right)\left(h_{\beta\gamma;\delta} + h_{\beta\delta;\gamma} - h_{\gamma\delta;\beta}\right)u^{\gamma}u^{\delta}$$

Introducing $\tilde{u}^{\mu}(x)$ which coincides with u^{μ} in the limit $x \to z(\tau)$

$$\tilde{u}_{\alpha;\beta} = \tilde{u}_{\beta;\alpha} \qquad K_{(\mu\nu;\rho)} = 0 \quad \Rightarrow \quad \frac{dQ}{d\tau} \approx \left[K^{\nu}{}_{\mu} \tilde{u}^{\mu} \partial_{\nu} \frac{\psi}{\Sigma} \right]_{x \to z(\lambda)}$$

$$\psi \equiv h_{\alpha\beta} \tilde{u}^{\alpha} \tilde{u}^{\beta} \Sigma$$

Or explicitly,

$$\frac{dQ}{d\tau} \approx \int d\lambda \left[\left(-\frac{P(r)}{\Delta}\left(\left(r^2 + a^2\right)\partial_t + a\partial_{\phi}\right) - \frac{dr}{d\lambda}\partial_r \right)\psi \right]_{x \to z(\lambda)}$$

∃ expression as simple as dE/dt by the balance argument?

Similarity between expressions for dE/dt and dQ/dt

- Energy loss can be also evaluated from the self-force.

$$\frac{dE}{d\tau} \approx \left[-\xi^{\nu}_{(t)} \partial_{\nu} \frac{\psi}{\Sigma} \right]_{x \to z(\lambda)} \Longleftrightarrow \frac{dQ}{d\tau} \approx \left[K^{\nu}{}_{\mu} \tilde{u}^{\mu} \partial_{\nu} \frac{\psi}{\Sigma} \right]_{x \to z(\lambda)}$$

- Formula obtained by the energy balance argument:

$$\frac{dE}{dt} \approx -\sum_{l,m,\omega} \left| Z_{l,m,\omega} \right|^2 \qquad \frac{dL}{dt} \approx -\sum_{l,m,\omega} \frac{m}{\omega} \left| Z_{l,m,\omega} \right|^2$$

$$Z_{l,m,\omega} \approx \int \left[\bar{\Pi}^{\mu\nu} T_{\mu\nu} \right]_{x \to z(\tau)} d\tau$$

- *dQ/dt* has a similar formula

$$\frac{dQ}{dt} \approx -\sum_{l,m,\omega} \hat{Z}_{l,m,\omega} \overline{Z_{l,m,\omega}}$$

$$\hat{Z}_{l,m,\omega} \approx \int \frac{d\tau}{-i\omega} K^{\rho}{}_{\sigma} u^{\sigma} \left[\partial_{\rho} \left(\Pi^{\mu\nu} T_{\mu\nu} \right) \right]_{x \to z(\tau)}$$

Further simplification

- A remarkable property of the Kerr geodesic equation

$$\left(\frac{dr}{d\lambda}\right)^2 = R(r) \qquad \left(\frac{d\theta}{d\lambda}\right)^2 = \Theta(\theta) \qquad d\lambda = d\tau / \Sigma$$ ··· Mino time

$$\Sigma = r^2 + a^2\cos^2\theta$$

r- and θ-oscillations can be solved independently

$$\frac{dt}{d\lambda} = -a\left(aE\sin^2\theta - L\right) + \frac{r^2+a^2}{\Delta}\left[E\left(r^2+a^2\right) - aL\right]$$

$$\frac{d\phi}{d\lambda} = -\left(aE - \frac{L}{\sin^2\theta}\right) + \frac{a}{\Delta}\left[E\left(r^2+a^2\right) - aL\right]$$

$$\Delta = r^2 - 2Mr + a^2$$

$$t(\lambda) = t^{(r)} + t^{(\theta)} + \left\langle \frac{dt}{d\lambda} \right\rangle \lambda$$ & similar expression for $\phi(\lambda)$

periodic functions with periods $2\pi\Omega_r^{-1}, 2\pi\Omega_\theta^{-1}$

⇨ Only discrete Fourier components arise:

$$\omega = \omega_m^{n_r, n_\theta} = \langle dt / d\lambda \rangle^{-1} \left(m \langle d\phi / d\lambda \rangle + n_r\Omega_r + n_\theta\Omega_\theta \right)$$

Final expression for dQ/dt

Sago, Tanaka, Hikida & Nakano ('05)

$$\left\langle \frac{dQ}{dt} \right\rangle = 2\left\langle \frac{(r^2+a^2)P(r)}{\Delta} \right\rangle \left\langle \frac{dE}{dt} \right\rangle - 2\left\langle \frac{aP(r)}{\Delta} \right\rangle \left\langle \frac{dL}{dt} \right\rangle + 2\sum_{l,m,n_r,n_\theta} \frac{n_r \Omega_r}{\omega} \left| Z_{l,m,\omega} \right|$$

$$\omega = \omega(m, n_r, n_\theta) \sim m\Omega_\phi + n_r \tilde{\Omega}_r + n_\theta \tilde{\Omega}_\theta$$

$$P(r) = E(r^2+a^2) - aL$$

orbital freq. in r & θ directions

This expression is as easy to evaluate as dE/dt and dL/dt.

$$\left\langle \frac{dE}{dt} \right\rangle = -\sum_{l,m,n_r,n_\theta} \left| Z_{l,m,\omega} \right|^2 \qquad \left\langle \frac{dL}{dt} \right\rangle = -\sum_{l,m,n_r,n_\theta} \frac{m}{\omega} \left| Z_{l,m,\omega} \right|^2$$

Analytic evaluation of dE/dt, dL/dt and dQ/dt for generic orbits has been done (Ganz et al., in prep.)

dQ/dt to $O(v^5 e^2 y)$

Sago et al. ('05)

$$
\begin{aligned}
\left\langle \frac{dQ}{dt} \right\rangle = & -\frac{64}{5}\mu^2 v^6 \\
& \times[1 - qv - \frac{743}{336}v^2 - (\frac{1637}{336}q - 4\pi)v^3 \\
& +(\frac{439}{48}q^2 - \frac{129193}{18144} - 4\pi q)v^4 + (\frac{151765}{18144}q - \frac{4159}{672}\pi - \frac{33}{16}q^3)v^5 \\
& +\{\frac{43}{8} - \frac{51}{8}qv - \frac{2425}{224}v^2 - (\frac{14869}{224}q - \frac{337}{8}\pi)v^3 \\
& -(\frac{453601}{4536} - \frac{3631}{32}q^2 + \frac{369}{8}\pi q)v^4 \\
& +(\frac{141049}{9072}q - \frac{38029}{672}\pi - \frac{929}{32}q^3)v^5\}e^2 \\
& +\{\frac{1}{2}qv + \frac{1637}{672}qv^3 - (\frac{1355}{96}q^2 - 2\pi q)v^4 \\
& -(\frac{151765}{36288}q - \frac{213}{32}q^3)v^5\}y \\
& +\{\frac{51}{16}qv + \frac{14869}{448}qv^3 + (\frac{369}{16}\pi q - \frac{33257}{192}q^2)v^4 \\
& +(-\frac{141049}{18144}q + \frac{5981}{64}q^3)v^5\}e^2 y]
\end{aligned}
$$

$v = \sqrt{M / r_0}$

r_0 ~ mean radius

e ~ eccentricity

$$q = \frac{a}{M}$$

$$y = \frac{C}{L_z^2} = \frac{Q - (aE - L_z)^2}{L_z^2} \quad \left(\sim \frac{L_x^2 + L_y^2}{L_z^2} \right)$$

Summary

- BH perturbation is a useful tool to investigate EMRIs.
- MiSaTaQuWa force describes local gravitational reaction to an orbiting particle, but its explicit evaluation seems very difficult to do.
- Under adiabatic approximation, rate of change of energy, angular momentum and Carter constant can be evaluated by GW amplitudes at infinity and horizon.