Gravitational Waves from Inspiralling Compact Binaries in Elliptical Orbits

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Based on

3PN Gravitational wave fluxes of energy and angular momentum from inspiralling eccentric binaries

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Introduction

- Inspiralling Compact Binaries (ICB) are considered to be the most probable sources of detectable gravitational radiation for laser interferometric gravitational-wave detectors. ICB are usually modeled as point particles in *quasi-circular* orbits.
- For long lived compact binaries, the quasi-circular approximation is quite appropriate. Gravitational Radiation Reaction (GRR) decreases the orbital eccentricity to negligible values by the epoch the emitted gravitational radiation enters the sensitive bandwidth of the interferometers. For an isolated binary, the eccentricity goes down roughly by a factor of three, when its semi-major axis is halved since $e/e_0 = (a/a_0)^{19/12}$ - (Peters, 64)

Eccentric Binaries

Based on Koenigdorffer and Gopakumar

- Stellar-mass compact binaries in eccentric orbits are excellent sources for LISA.
- LISA will "hear" GW from intermediate-mass black holes moving in highly eccentric orbits

  K. Gültekin, M. C. Miller, and D. P. Hamilton (2005), T. Matsubayashi, J. Makino, and T. Ebisuzaki (2005), M. A. Gürkan, J. M. Fregeau, and F. A. Rasio (2005)
- Several papers indicate that SMBHB formed from galactic mergers, may coalesce with orbital eccentricity

  S. J. Aarseth (2003), P. Berczik, D. Merritt, R. Spurzem, and H.-P. Bischof (2006), O. Blaes, M. H. Lee, and A. Socrates(2002), P. J. Armitage and P. Natarajan (2005), M. Iwasawa, Y. Funato, and J. Makino (2005)

  These investigations employ different techniques and astrophysical scenarios to reach the above conclusion.

Kozai Mechanism

- One proposed astrophysical scenario, involves hierarchical triplets modeled to consist of an inner and an outer binary. If the mutual inclination angle between the orbital planes of the inner and of the outer binary is large enough, then the time averaged tidal force on the inner binary may induce oscillations in its eccentricity, known in the literature as the Kozai mechanism

Kozai (1962),M. C. Miller and D. P. Hamilton (2002), E. B. Ford, B. Kozinsky, and F. A. Rasio (2000), Wen (2003)

Kozai Mechanism, Globular Clusters

- In globular clusters (GC), the inner binaries of hierarchical triplets undergoing Kozai oscillations can merge under GRR

  M. C. Miller and D. P. Hamilton (2002).

  A good fraction of such systems will have eccentricity~ 0.1, when emitted GW from these binaries passes through $10\,\mathrm{Hz}$

  Wen (2003).

- Such scenarios involving compact *eccentric* binaries are being suggested as potential GW sources for the terrestrial GW detectors.

NS-BH...GRB

- During the late stages of BH–NS inspiral the binary can become eccentric

  M. B. Davies, A. J. Levan, and A. R. King (2005).

  In general NS is not disrupted at the first phase of mass transfer and what remains of NS is left on a wider eccentric orbit from where it again inspirals back to the black hole. Scenario invoked to explain the light curve of the short gamma-ray burst GRB 050911

  Page (2006)

- At least partly short GRBs are produced by the merger of NS–NS binaries, formed in GC by exchange interactions involving compact objects

  J. Grindlay, S. P. Zwart, and S. McMillan, (2006)

  A distinct feature of such binaries is that they have high eccentricities at short orbital separation.

Kicks, Eccentricity

- Compact binaries that merge with some residual eccentricities may be present in galaxies too. Chaurasia and Bailes demonstrated that a natural consequence of an asymmetric kick imparted to neutron stars at birth is that the majority of NS–NS binaries should possess highly eccentric orbits

  H. K. Chaurasia and M. Bailes (2005).

- Observed deficit of highly eccentric short-period binary pulsars was attributed to selection effects in pulsar surveys.
- Conclusions are applicable to BH–NS and BH–BH binaries.

Compact star clusters

- Yet another scenario that can create inspiralling eccentric binaries with short periods involves compact star clusters. It was noted that the interplay between GW-induced dissipation and stellar scattering in the presence of an intermediate-mass black hole can create short-period highly eccentric binaries

  C. Hopman and T. Alexander (2005)

- A very recent attempt to model realistically compact clusters that are likely to be present in galactic centers indicates that compact binaries usually merge with eccentricities

  G. Kupi, P. Amaro-Seoane, and R. Spurzem (2006),

Implication of Eccentricity for GWDA

Martel Poisson

- Investigated reduction in SNR if eccentric signals are recd but searched for in data by circular templates - nonoptimal signal processing
- Found that for a binary system of given total mass, the *loss increases* with *increasing eccentricity*
- For a given eccentricity, *loss decreases* as *total mass is increased*

FF as fn of initial eccentricity e_0

Martel and Poisson

e_0	$1.0+1.0$	$1.4+1.4$	$1.4+2.5$	$1.4+5.0$	$1.4+10$	$3.0+6.0$	$6.0+6.0$
.00	0.998	0.997	0.999	0.998	0.998	0.998	0.998
.05	0.960	0.976	0.985	0.992	0.992	0.998	0.996
.10	0.898	0.931	0.947	0.965	0.976	0.984	0.993
.15	0.836	0.879	0.902	0.930	0.946	0.961	0.975
.20	0.762	0.822	0.854	0.893	0.913	0.934	0.955
.25	0.695	0.761	0.802	0.852	0.885	0.903	0.930
.30	0.630	0.637	0.749	0.805	0.850	0.868	0.900
.35	0.569	0.581	0.693	0.753	0.811	0.829	0.867
.40	0.513	0.520	0.635	0.698	0.765	0.783	0.827
.45	0.454	0.460	0.574	0.637	0.714	0.732	0.781
.50	0.397	0.402	0.513	0.576	0.656	0.675	0.728
.55	0.348	0.350	0.452	0.513	0.595	0.614	–
.60	0.297	0.303	0.396	0.452	0.534	–	–
.65	0.257	0.231	0.344	–	–	–	–

Present Work

- For binaries moving in general orbits, we compute *all* the instantaneous contributions to the 3PN accurate GW energy and angular momentum flux.
- For binaries moving in elliptical orbits, hereditary terms are computed exploiting the double periodicity of the PN motion
- Flux averaged over an elliptical orbit using 3PN quasi-Keplerian parametrization of the binary's orbital motion by Memmesheimer, Gopakumar and Schäfer
- Complete expressions for the far-zone energy flux from inspiralling compact binaries moving in eccentric orbits.
- Compute evolution of orbital elements a_r, e_t, n under 3PN Grav Radn reaction (5.5PN terms in accn)

Present Work

- Represent GW from a binary evolving negligibly under GRR including precisely upto 3PN order, the effects of eccentricity and periastron precession during epochs of inspiral when the orbital parameters are essentially constant over a few orbital revolutions.
- First step towards the discussion of the *quasi-elliptical* case: the evolution of the binary in an elliptical orbit under GRR

The Generation Modules

- Generation problem for GW at any PN order requires solution to two independent problems
- First relates to the equation of motion of the binary
- 3PN EOM for ICB on general orbits are complete (Jaranowski, Schäfer; Blanchet, Faye; Damour, Jaranowski, Schäfer; Blanchet, Damour, Esposito-Farèse);
- Second to FZ fluxes of energy, angular momentum (Present Work)
- Latter requires the computation of the relativistic mass and current multipole moments to appropriate PN orders.
- Relevant multipoles including the 3PN Mass Quadrupole complete for general orbits
  Blanchet, Damour, Iyer, Esposito-Farèse; Present work
- Unlike at earlier PN orders, the 3PN contribution to energy flux come not only from the 'instantaneous' terms but also include 'hereditary' contributions arising from the *tail of tails* and *tail-square* terms.

FZ flux - Radiative Multipoles

Following Thorne (1980), the expression for the 3PN accurate far zone energy flux in terms of symmetric trace-free (STF) radiative multipole moments read as

$$
\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)_{\mathrm{far-zone}} = \quad & \frac{G}{c^5}\left\{\frac{1}{5}U^{(1)}_{ij}U^{(1)}_{ij}\right. \\
& +\frac{1}{c^2}\left[\frac{1}{189}U^{(1)}_{ijk}U^{(1)}_{ijk}+\frac{16}{45}V^{(1)}_{ij}V^{(1)}_{ij}\right] \\
& +\frac{1}{c^4}\left[\frac{1}{9072}U^{(1)}_{ijkm}U^{(1)}_{ijkm}+\frac{1}{84}V^{(1)}_{ijk}V^{(1)}_{ijk}\right] \\
& +\frac{1}{c^6}\left[\frac{1}{594000}U^{(1)}_{ijkmn}U^{(1)}_{ijkmn}+\frac{4}{14175}V^{(1)}_{ijkm}V^{(1)}_{ijkm}\right] \\
& \left.+\mathcal{O}(8)\right\}.
\end{aligned}
$$

PN order of Multipoles

- For a given PN order only a finite number of Multipoles contribute
- At a given PN order the mass l-multipole is accompanied by the current $l-1$-multipole (Recall EM)
- To go to a higher PN order Flux requires new higher order l-multipoles and more importantly higher PN accuracy in the known multipoles.
- 3PN Energy flux requires 3PN accurate Mass Quadrupole, 2PN accurate Mass Octupole, 2PN accurate Current Quadrupole,........ N Mass 2^5-pole, Current 2^4-pole

Radiative moments - Source moments

The relations connecting the different radiative moments U_L and V_L to the corresponding source moments I_L and J_L are given below. For the mass type moments we have (Blanchet 92.. 98)

$$
\begin{aligned}
U_{ij}(U) \quad = \quad & I^{(2)}_{ij}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{2}\right] I^{(4)}_{ij}(U-\tau) \\
& + \frac{G}{c^5}\left\{ -\frac{2}{7}\int_0^{+\infty} d\tau I^{(3)}_{a<i}(U-\tau) I^{(3)}_{j>a}(U-\tau) \right. \\
& \qquad + \frac{1}{7} I^{(5)}_{a<i} I_{j>a} - \frac{5}{7} I^{(4)}_{a<i} I^{(1)}_{j>a} - \frac{2}{7} I^{(3)}_{a<i} I^{(2)}_{j>a} + \frac{1}{3}\varepsilon_{ab<i} I^{(4)}_{j>a} J_b \\
& \qquad \left. + 4\left[W^{(2)} I_{ij} - W^{(1)} I^{(1)}_{ij}\right]\right\} \\
& + 2\left(\frac{GM}{c^3}\right)^2 \int_0^{+\infty} d\tau I^{(5)}_{ij}(U-\tau) \\
& \left[\ln^2\left(\frac{c\tau}{2r_0}\right) + \frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{124627}{44100}\right] \\
& + \mathcal{O}(7),
\end{aligned}
$$

Radiative moments - Source moments

$$
\begin{aligned}
U_{ijk}(U) &= I^{(3)}_{ijk}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{97}{60}\right] I^{(5)}_{ijk}(U-\tau) \\
&\quad + \mathcal{O}(5)\,, \\
U_{ijkm}(U) &= I^{(4)}_{ijkm}(U) + \frac{G}{c^3}\left\{2M\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{59}{30}\right] I^{(6)}_{ijkm}(U-\tau)\right. \\
&\qquad + \frac{2}{5}\int_0^{+\infty} d\tau I^{(3)}_{<ij}(U-\tau) I^{(3)}_{km>}(U-\tau) \\
&\qquad \left. - \frac{21}{5} I^{(5)}_{<ij} I_{km>} - \frac{63}{5} I^{(4)}_{<ij} I^{(1)}_{km>} - \frac{102}{5} I^{(3)}_{<ij} I^{(2)}_{km>}\right\} + \mathcal{O}(4)\,,
\end{aligned}
$$

Requires one to control the Reln of the Radiative Mass Quadrupole to Source Mass Quadrupole to 3PN accuracy. Hence involves *Tail-of-Tails* for Mass Quadrupole. Other multipoles to lower PN accuracy involving only *Tails*

Current-type moments

$$
\begin{aligned}
V_{ij}(U) &= J^{(2)}_{ij}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{7}{6}\right] J^{(4)}_{ij}(U-\tau) \\
&\quad +\mathcal{O}(5)\,, \\
V_{ijk}(U) &= J^{(3)}_{ijk}(U) + \frac{G}{c^3}\left\{2M\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{5}{3}\right] J^{(5)}_{ijk}(U-\tau)\right. \\
&\qquad \left. + \frac{1}{10}\varepsilon_{ab<i} I^{(5)}_{j\underline{a}} I_{k>b} - \frac{1}{2}\varepsilon_{ab<i} I^{(4)}_{j\underline{a}} I^{(1)}_{k>b} - 2J_{<i} I^{(4)}_{jk>}\right\} \\
&\quad +\mathcal{O}(4)\,.
\end{aligned}
$$

$$
\begin{aligned}
U_L(U) &= I^{(l)}_L(U) + \mathcal{O}(3)\,, \\
V_L(U) &= J^{(l)}_L(U) + \mathcal{O}(3)\,.
\end{aligned}
$$

$$
U = t - \frac{\rho}{c} - \frac{2\,G\,M}{c^3}\ln\left(\frac{\rho}{c\,r_0}\right)\,.
$$

Instantaneous Terms

$$\left(\frac{d\mathcal{E}}{dt}\right) = \left(\frac{d\mathcal{E}}{dt}\right)_{\mathrm{inst}} + \left(\frac{d\mathcal{E}}{dt}\right)_{\mathrm{hered}} .$$

$$\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)_{\mathrm{inst}} = \; & \frac{G}{c^5}\left\{ \frac{1}{5} I_{ij}^{(3)} I_{ij}^{(3)} \right. \\
& + \frac{1}{c^2}\left[\frac{1}{189} I_{ijk}^{(4)} I_{ijk}^{(4)} + \frac{16}{45} J_{ij}^{(3)} J_{ij}^{(3)}\right] + \frac{1}{c^4}\left[\frac{1}{9072} I_{ijkm}^{(5)} I_{ijkm}^{(5)} + \frac{1}{84} J_{ijk}^{(4)} J_{ijk}^{(4)}\right] \\
& + \frac{8G}{5c^5}\left\{ I_{ij}^{(3)} \left[I_{ij} W^{(5)} + 2 I_{ij}^{(1)} W^{(4)} - 2 I_{ij}^{(3)} W^{(2)} - I_{ij}^{(4)} W^{(1)} \right]\right\} \\
& + \frac{2G}{5c^5} I_{ij}^{(3)} \left\{ -\frac{4}{7} I_{ai}^{(5)} I_{aj}^{(1)} - I_{ai}^{(4)} I_{aj}^{(2)} - \frac{4}{7} I_{ai}^{(3)} I_{aj}^{(3)} + \frac{1}{7} I_{ai}^{(6)} I_{aj} + \right. \\
& \left. \frac{1}{3} \epsilon_{abi} \left(I_{aj}^{(4)} J_b^{(1)} + I_{aj}^{(5)} J_b \right) \right\} \\
& \left. + \frac{1}{c^6}\left[\frac{1}{594000} I_{ijkmn}^{(6)} I_{ijkmn}^{(6)} + \frac{4}{14175} J_{ijkm}^{(5)} J_{ijkm}^{(5)}\right] + \mathcal{O}(8) \right\} .
\end{aligned}$$

Tail terms in the 3PN energy flux

- Three kinds of hereditary terms appear in the computation.
- The 'tails' coming from the multipole interaction of the mass quadrupole with the ADM mass ($M \times I_{ij}$),
- 'Tails of tails' due to the cubic nonlinear interaction $M \times M \times I_{ij}$
- Tail-squared term arising from quadrupole-quadrupole interaction $I_{ij} \times I_{kl}$.
- The hereditary terms in the energy flux can be written as

Tail terms in the 3PN energy flux

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\text{hered}} = \left(\frac{d\mathcal{E}}{dt}\right)_{\text{tail}} + \left(\frac{d\mathcal{E}}{dt}\right)_{\text{tail(tail)}} + \left(\frac{d\mathcal{E}}{dt}\right)_{(\text{tail})^2}.$$

$$\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)_{\text{tail}} = \frac{4G^2M}{c^5}\Bigg\{ & \frac{1}{5c^3} M_{ij}^{(3)} \int_0^{+\infty} d\tau M_{ij}^{(5)}(T_R-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{11}{12}\right] \\
& + \frac{1}{189c^5}\, M_{ijk}^{(4)} \int_0^{+\infty} d\tau M_{ijk}^{(6)}(T_R-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{97}{60}\right] \\
& + \frac{16}{45c^5}\, S_{ij}^{(3)} \int_0^{+\infty} d\tau\, S_{ij}^{(5)}(T_R-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{7}{6}\right] \\
& + \frac{1}{9072c^7}\, M_{ijkl}^{(5)} \int_0^{+\infty} d\tau\, M_{ijkl}^{(7)}(T_R-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{59}{30}\right] \\
& + \frac{1}{84c^7}\, S_{ijk}^{(4)} \int_0^{+\infty} d\tau\, S_{ijk}^{(6)}(T_R-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{5}{3}\right] \\
& + O\left(\frac{1}{c^8}\right)\Bigg\},
\end{aligned}$$

Tail terms in the 3PN energy flux

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\mathrm{tail(tail)}} = \frac{4G^2M}{c^5}\left\{\frac{GM}{5c^6}M_{ij}^{(3)}\int_0^{+\infty}d\tau M_{ij}^{(6)}(T_R-\tau)\right.$$
$$\left.\times\left[\ln^2\left(\frac{c\tau}{2r_0}\right)+\frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right)+\frac{124627}{44100}\right]+O\left(\frac{1}{c^8}\right)\right\},$$
$$\left(\frac{d\mathcal{E}}{dt}\right)_{(\mathrm{tail})^2} = \frac{4G^2M}{c^5}\left\{\frac{GM}{5c^6}\left(\int_0^{+\infty}d\tau M_{ij}^{(5)}(T_R-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{11}{12}\right]\right)^2\right.$$
$$\left.+\,O\left(\frac{1}{c^{10}}\right)\right\}.$$

Constant scaling the logarithm has been chosen to be r_0 to match with the choice made in the computation of tails-of-tails in (Blanchet, 98). It is a freely specifiable constant, entering the relation between the retarded time $U = T - R/c$ in radiative coordinates and the corresponding time $t - \rho/c$ in harmonic coordinates (where ρ is the distance of the source in harmonic coordinates). More precisely we have

$$U = t - \frac{\rho}{c} - \frac{2\,G\,M}{c^3}\ln\left(\frac{\rho}{c\,r_0}\right).$$

Hereditary Contributions

- Multipole moments describing GW emitted by an isolated system cannot evolve independently. They couple to each other and with themselves, giving rise to non-linear physical effects.
- Instantaneous terms in the flux must be supplemented by the contributions arising from these non-linear multipole interactions.
- Leading multipole interaction is between the mass quadrupole moment M_{ij} and the mass monopole M or ADM mass. Associated with the non-linear effect of tails at order 1.5PN.
- Physically due to the backscatter of linear waves from ST curvature generated by the mass monopole M.

Hereditary Contributions

- Detailed study of tails is due to Blanchet (1998) based on the MPM formalism of Blanchet and Damour. Showed that up to 3PN these comprise the dominant quadratic order tails, the cubic-order tails or tails of tails and the non-linear memory integral

  Christodoulu 91, Will- Wiseman (1991), Thorne 92, Arun et al 04.

- We set up a general theoretical framework to compute the hereditary contributions for binaries moving in elliptical orbits and apply it to evaluate all the tail contributions contained in the 3PN accurate GW energy flux.

Instantaneous vs Hereditary Contributions

- For instantaneous terms in the energy flux, explicit closed form analytical expressions can be given in terms of dynamical variables related to relative speed v and relative separation r. These expressions can be conveniently averaged in the time domain over an orbit using its quasi-Keplerian representation.
- For hereditary contribution one can only write down formal analytical expressions as integrals over the past. More explicit expressions in terms of the dynamical variables require in addition a *model* of the binary's orbit to implement the integration over the past history.

Hereditary Contributions - Circular orbits

- In the circular orbit case, with a simplified model of binary inspiral one can work directly in the time domain. Eg Blanchet (1998) computed the hereditary terms in the flux upto 3.5PN while Arun, Blanchet, Iyer and Qusailah (2004) evaluated the GW polarisations upto 2.5PN.
- The tail integrals are evaluated using standard integrals for a *fixed* non-decaying circular orbit. 'Remote-past' contribution to the tail integrals can be proved to be negligible and errors due to inspiral by gravitation radiation reaction to be at least 4PN.

Hereditary Contributions - Elliptical Orbits

1.5PN

- In the elliptic orbit case situation is more involved. Even after using the quasi-Keplerian parametrization, one cannot perform the integrals in the time domain (as for the circular orbit case), since the multipole moments have a more complicated dependence on time so that the integrals are not analytically solvable in simple closed forms.
- By working in the Fourier domain to explicitly evaluate the hereditary integrals, Blanchet and Schäfer (1993) computed the hereditary tail terms at 1.5PN for elliptical orbits using the lowest order Newtonian Keplerian representation.

Blanchet-Damour 88, Blanchet-Schäfer 93

- GW tail only slightly sensitive to detailed dynamics of source at early times under weak assumption of *moderation* of wave emission in the past
- Integrand of wave tail contains a log kernel that blows up at past infinity and hence requires assumption on the source at very early times..Eg second time derivative of MQ becomes constant.. precludes a strong burst
- Under the above assumptions, by splitting the tail integral into a 'remote past' and 'recent past' integrals, using the $1/x$ ($x = (U - V)/T$) fall-off of the kernel in the remote past, Blanchet and Schäfer obtain a convergent integal for each Fourier component, independent of the constant T used to split the integral.

Blanchet-Damour 88, Blanchet-Schäfer 93

- They then show, an important simplification is the possibility to use for the tail computation a fixed orbit rather than a decaying orbit (a Fourier series rather than a Fourier transform)
- Since the remote past contribution to the wave tail is small, by choosing T to be the current value of the coalescing time, only the recent past contribution is relevant
- They also derive the same result using a different adiabatic damping regularisation procedure
- Explicitly show wave field for a decaying orbit can be obtained by formal replacement of wave field at constant frequency ω_0 by varying frequency $\omega(U)$ and similarly for phase (in limit $2\pi/\omega_0 T \to 0$)

Hereditary Terms - Elliptic Orbits (2.5PN and 3PN)

1PN Quasi-Keplerian Reprn

- To tackle terms at 2.5PN and 3PN we need to go beyond the (Newtonian) Keplerian representation to a 1PN quasi-Keplerian representation of the orbit Damour and Deruelle (85,86). Then r and ℓ are expressed in terms of the eccentric anomaly u as

$$\begin{aligned} r &= a_r(1 - e_r \cos u)\,, \\ \ell &= u - e_t \sin u. \end{aligned}$$

- The phase angle ϕ is given by

$$\phi = KV,$$

where the true anomaly V is defined by,

$$V = 2 \arctan\left[\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2} \tan\frac{u}{2}\right].$$

- The possible additive constant in the equation for ϕ is set equal to zero.

Quasi-Keplerian Reprn

- K is the periastron advance such that the precession of the periastron per period is $\Delta\phi = 2\pi(K-1)$. As K tends to one in the limit $c \to \infty$ (Newtonian limits), it is convenient to introduce $k \equiv K - 1$, to describe the *relativistic precession*.
- 1PN parametrization of the binary involves three kinds of eccentricities (e_r, e_t and e_ϕ) which makes the algebra more involved.
- More seriously at 1PN order, the periastron precession effect appears in the problem and one has to contend with two times scales: the orbital time scale and the periastron precession time scale.

Doubly-periodic structure of the solution

- The phase is written as as

$$\phi = \phi_{\mathrm{P}} + K\ell + W(\ell),$$

- Periodic function $W(\ell)$ reads

$$W = K\,(V - \ell)\ .$$

- By *doubly periodic* one means that
  the radial motion $r(t)$ is periodic with period P
  while the angular motion $\phi(t)$ is periodic (modulo 2π) with a **different** period P/k.
- Only when the two periods are commensurable i.e. $k = 1/N$ where N is a natural number is the motion *periodic* in space (i.e. the orbit in space closed)

Hereditary Terms - Elliptic Orbits - Procedure

- Express all the multipole moments needed for the hereditary computation at Newtonian order as discrete Fourier series in l. For moments needed beyond the lowest Newtonian order the double periodicity needs to be crucially incorporated.
- Evaluation of the Fourier coefficients is done either numerically or in terms of an infinite sum of combinations of Bessel functions.
- All tail terms at 2.5PN and 3PN are completely computed to provide the 'enhancement factors' for binaries in elliptical orbits at the 2.5PN and 3PN orders, extending classic work of Peters and Mathews

Fourier decomposition of Mult Moments

- Multipole moments of the CB system will be denoted by $I_L(t)$ (mass-type moment) and $J_L(t)$ (current-type)
- General structure of these mass and current moments, I_L and say J_{L-1} (where $L-1$ is taken rather than L for convenience), at any post-Newtonian order for a binary system moving on a general non-circular orbit is of the type

$$
\begin{aligned}
I_L(t) &= \sum_{p=0}^{l} \mathcal{F}_p[r, \dot{r}^2, v^2]\, x^{<i_1\cdots i_p} v^{i_{p+1}\cdots i_l>}, \\
J_{L-1}(t) &= \sum_{p=0}^{l-2} \mathcal{G}_p[r, \dot{r}^2, v^2]\, x^{<i_1\cdots i_p} v^{i_{p+1}\cdots i_{l-2}} \varepsilon^{i_{l-1}>ab} x^a v^b,
\end{aligned}
$$

where x^i and v^i are the relative position and velocity of the two bodies, and the coefficients $\mathcal{F}_p$ and $\mathcal{G}_p$ depend on r, $\dot{r}^2$ and $v^2 = \dot{r}^2 + r^2\dot{\phi}^2$.

Fourier decomposition of Mult Moments

- For quasi-elliptic motion in a plane, inserting $x = r\cos\phi$, $y = r\sin\phi$, and $v_x = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$, $v_y = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$, we can explicitly factorize out the dependence on the orbital phase ϕ.
- Using the explicit solution of the motion one can express r, $\dot{r}^2$ and v^2, and hence the $\mathcal{F}_p$'s and $\mathcal{G}_p$'s, as periodic function of the mean anomaly $\ell = n(t - t_0)$, where $n = 2\pi/P$.
- General structure can be expressed in terms of the phase angle ϕ, as the following finite sum over an index m ranging from $-l$ to $+l$,

$$\begin{aligned} I_L(t) &= \sum_{m=-l}^{l} {}_{(m)}\mathcal{A}_L(\ell)\, e^{im\phi}, \\ J_{L-1}(t) &= \sum_{m=-l}^{l} {}_{(m)}\mathcal{B}_{L-1}(\ell)\, e^{im\phi}, \end{aligned}$$

with some complex coefficients ${}_{(m)}\mathcal{A}_L$ and ${}_{(m)}\mathcal{B}_{L-1}$.

Fourier decomposition of Mult Moments

- Important point for our purpose is that the functions ${}_{(m)}\mathcal{A}_L(\ell)$ and ${}_{(m)}\mathcal{B}_{L-1}(\ell)$ are now 2π-periodic functions of ℓ.
- Structure of the mass and current moments is the same, but the coefficients ${}_{(m)}\mathcal{A}_L$ and ${}_{(m)}\mathcal{B}_{L-1}$ have a different parity, because of the Levi-Civita symbol ε^{iab} entering the current moment.
- In above expressions, the variable ϕ is not a periodic function of ℓ.
- To proceed further, we need to exploit the double periodicity of the dynamics in the two variables $\lambda = K\ell$ and ℓ by writing, $\phi = K\ell + W(\ell)$, where $W(\ell)$ is periodic in ℓ
- More convenient to single out in the expression of the phase the purely relativistic precession of the periastron $k\ell$, where $k = K - 1$. This yields many factors which will modify the coefficients but in such a way that they remain periodic in ℓ. Hence we can write

Fourier decomposition of Mult Moments

$$I_L(t) = \sum_{m=-l}^{l} \underset{(m)}{\mathcal{I}}{}_L(\ell)\, e^{imk\ell},$$

$$J_{L-1}(t) = \sum_{m=-l}^{l} \underset{(m)}{\mathcal{J}}{}_{L-1}(\ell)\, e^{imk\ell}.$$

- This makes it possible to use a Fourier series expansion in the interval $[0, 2\pi]$ for each of the functions above leading then to the following discrete Fourier decompositions,

$$I_L(t) = \sum_{p=-\infty}^{+\infty} \sum_{m=-l}^{l} \underset{(p,m)}{\mathcal{I}}{}_L\, e^{i(p+mk)\ell},$$

$$J_{L-1}(t) = \sum_{p=-\infty}^{+\infty} \sum_{m=-l}^{l} \underset{(p,m)}{\mathcal{J}}{}_{L-1}\, e^{i(p+mk)\ell}.$$

Fourier decomposition of Mult Moments

- We compute all the tail and tail-of-tail terms in the averaged GW energy flux up to the 3PN order.

$$\langle \mathcal{F} \rangle \equiv -\langle \left(\frac{d\mathcal{E}}{dt} \right)^{\mathrm{GW}} \rangle \equiv -\langle \left(\int d\Omega \, \frac{d\mathcal{E}}{dt \, d\Omega} \right)^{\mathrm{GW}} \rangle,$$

- Together with the instantaneous terms one obtains complete expressions of the 3PN energy flux.
- Tails are not just mathematical curiosities in general relativity but facets that should show up in the GW signals of ICB and subsequently decoded by the GW detectors like VIRGO, LIGO and LISA

Eccentricity enhancement factors

- Define next some functions of the eccentricity by certain Fourier series of the components of the *Newtonian* moments $I_L = \mu x^{<L>}$ and $J_{L-1} = \mu x^{<L-2}\varepsilon^{i_{l-1}>ab}x^a v^b$ for a Keplerian ellipse with semi-major axis a, eccentricity e and frequency $n = 2\pi/P$. Rescale the moments to adimensionalize them by defining

$$\begin{aligned}\hat{I}_L &\equiv \frac{I_L^{(N)}}{\mu a^l},\\ \hat{J}_{L-1} &\equiv \frac{J_{L-1}^{(N)}}{\mu a^l\, n}.\end{aligned}$$

- Define some dimensionless Fourier series, which are functions only of the (Keplerian) eccentricity e. First

$$f(e) = \frac{1}{16}\sum_{p=1}^{+\infty} p^6 |\underset{(p)}{\hat{\mathcal{I}}}_{ij}|^2,$$

Eccentricity enhancement factors

- Peters & Mathews "enhancement" function, entering the energy flux at the Newtonian order (given by the Einstein quadrupole formula), *i.e.*

$$\mathcal{F}_{\mathrm{N}} = \frac{32}{5}\,\nu^2\,x^5\,f(e),$$

  when computed using Fourier series. Remarkably $f(e)$ admits an algebraically closed-form expression, crucial for the timing of the binary pulsar PSR 1913+16,

$$f(e) = \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}}.$$

- $f(e)$ is referred to as enhancement function since in the case of the binary pulsar, with eccentricity $e = 0.617$ it enhances the effect of the orbital $\dot{P}$ by a factor ~ 11.843

Eccentricity enhancement factors

- Define several other eccentricity "enhancement" functions which constitute useful ingredients when parametrizing the tail terms at Newtonian order. Introduce,

$$\begin{aligned}
\varphi(e) &= \frac{1}{32}\sum_{p=1}^{+\infty} p^7 |\underset{(p)}{\hat{\mathcal{I}}}_{ij}|^2, \\
\beta(e) &= \frac{20}{49209}\sum_{p=1}^{+\infty} p^9 |\underset{(p)}{\hat{\mathcal{I}}}_{ijk}|^2, \\
\gamma(e) &= 4\sum_{p=1}^{+\infty} p^7 |\underset{(p)}{\hat{\mathcal{J}}}_{ij}|^2.
\end{aligned}$$

- Like for $f(e)$ these functions are defined s.t. they tend to one in the circular orbit limit, when $e \to 0$. Unlike for $f(e)$, they do *not* admit closed-form expressions, and must be left in the form of Fourier series.

Eccentricity enhancement factors

- Function $\varphi(e)$ has already been computed numerically in Blanchet Schäfer (1993).
- Note $\varphi^{\rm BS}(e) = \varphi(e)/f(e)$. Here more convenient *not* to rescale the various functions using the Peters & Mathews "enhancement" function $f(e)$. In their terms Newtonian tail terms read

$$\begin{aligned}
\mathcal{F}_{\rm mass\ quad} &= \frac{32}{5}\,\nu^2\,x^{13/2}\left\{4\pi\,\varphi(e)\right\},\\
\mathcal{F}_{\rm mass\ oct} &= \frac{32}{5}\,\nu^2\,x^{15/2}\left\{\frac{16403}{2016}\,\pi\,\beta(e)\right\}(1-4\,\nu),\\
\mathcal{F}_{\rm curr\ quad} &= \frac{32}{5}\,\nu^2\,x^{15/2}\left\{\frac{1}{18}\pi\,\gamma(e)\right\}(1-4\,\nu),
\end{aligned}$$

Coefficient appropriate to the Newtonian expression of the flux for circular orbits is factored out

Eccentricity enhancement factors

- Introduce two other enhancement functions which are helpful when parametrizing the tail-of-tail and tail squared integrals (Newtonian in the present approximation). Namely

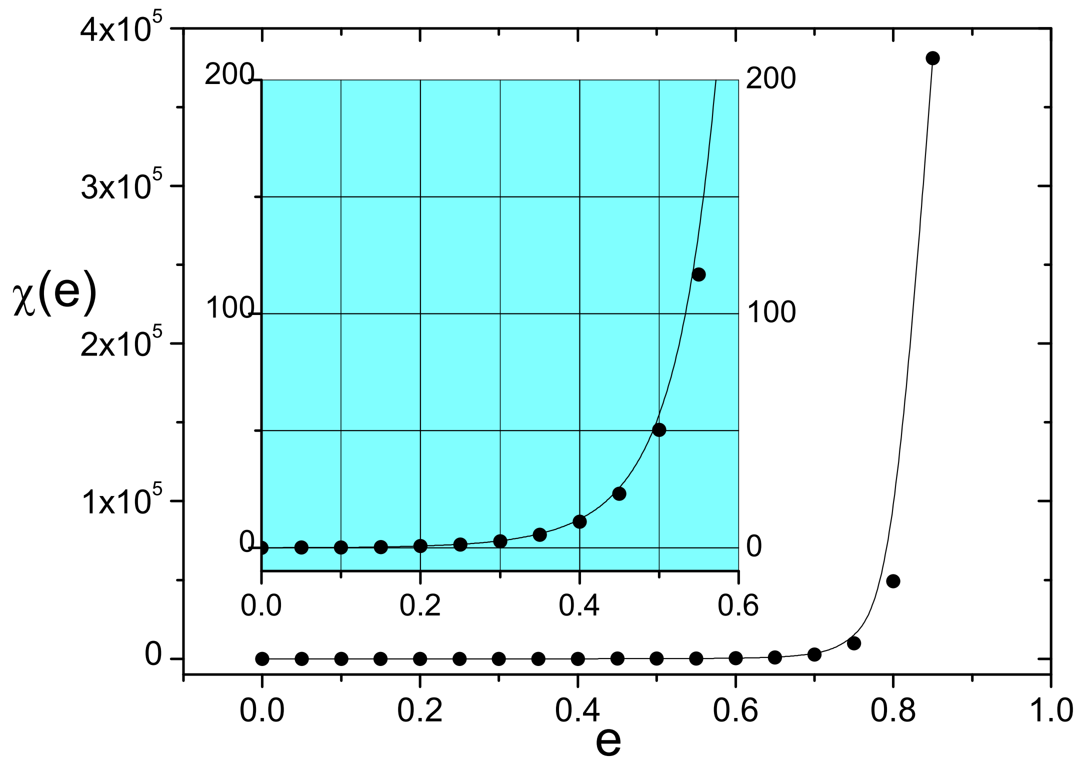
$$\begin{aligned} F(e) &= \frac{1}{64}\sum_{p=1}^{+\infty} p^8 |\underset{(p)}{\hat{\mathcal{I}}}_{ij}|^2, \\ \chi(e) &= \frac{1}{64}\sum_{p=1}^{+\infty} p^8 \ln\left(\frac{p}{2}\right) |\underset{(p)}{\hat{\mathcal{I}}}_{ij}|^2. \end{aligned}$$

- Easily checked by a straightforward calculation *à la* Peters & Mathews 63, that the function $F(e)$ admits an analytic form similar to the one of $f(e)$ and given by

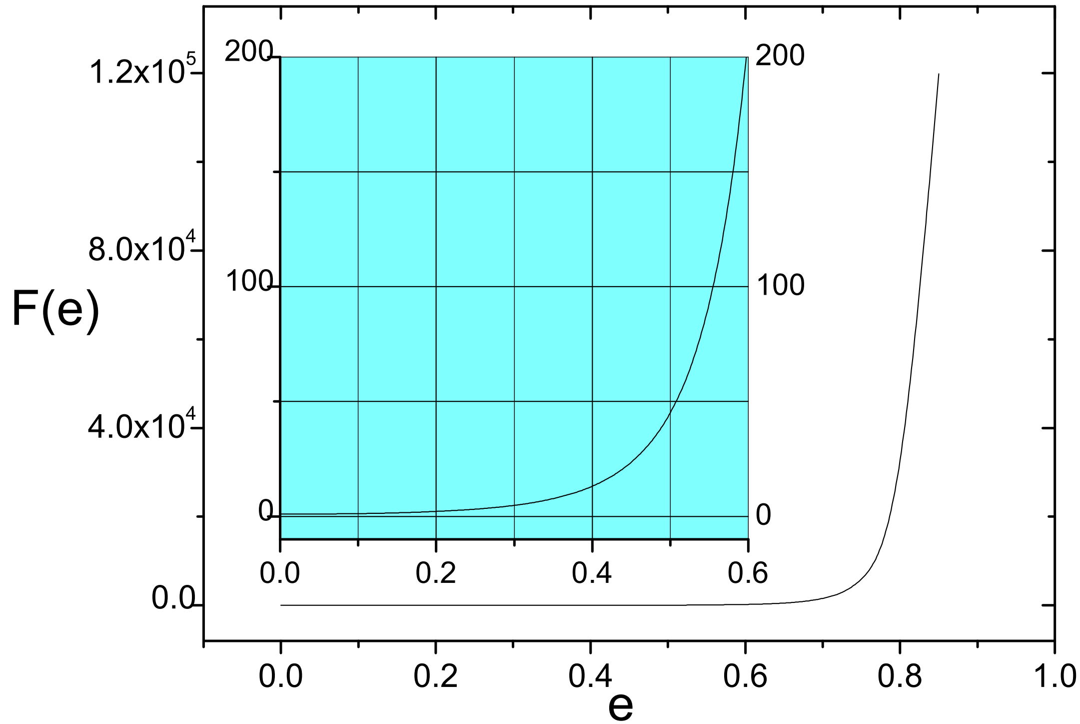
$$F(e) = \frac{1 + \frac{85}{6}e^2 + \frac{5171}{192}e^4 + \frac{1751}{192}e^6 + \frac{297}{1024}e^8}{(1-e^2)^{13/2}}.$$

Eccentricity enhancement factors -$\chi(e)$

- $\chi(e)$ does *not* admit any analytic form, but is easily seen to tend to zero when $e \to 0$.
- At Newtonian order and in the circular orbit limit, the quadrupole moment admits only one harmonic, which is the one for which $p = 2$. Because of the logarithmic term in $\chi(e)$, we see that the function is zero when $e = 0$.



Enhancement Function $F(e)$



Eccentricity enhancement factors

- Thes figures shows the variation of $\chi(e_t)$ and $F(e_t)$ with the eccentricity e_t. The inset plot is the zoom for the functions which look as straight horizontal lines in main graph. In the top panel the points are the numerical computation for $\chi(e_t)$ at e_t =0, 0.5, 1, 1.5,.... The solid lines are the fit to the numerical points. In the bottom panel, the exact function of $F(e_t)$ is used. For the circular limit, $e_t = 0$, $\chi(0) = 0$, $F(0) = 1$.
- In terms of $F(e)$ and $\chi(e)$, the sum of tail-of-tail and tail squared contributions computed earlier reads

$$\begin{aligned}\mathcal{F}_{\mathrm{tail(tail)+(tail)^2}} &= \frac{32}{5}\,\nu^2\,x^8\Bigg\{\Bigg[-\frac{116761}{3675}+\frac{16}{3}\pi^2-\frac{1712}{105}C \\ &\quad -\frac{1712}{105}\ln\left(\frac{4\omega r_0}{c}\right)\Bigg]F(e)-\frac{1712}{105}\,\chi(e)\Bigg\}.\end{aligned}$$

Eccentricity enhancement factors

- Finally for the 1PN quadrupole tail the calculation is much more complicated, as the Fourier series involve several summations. In addition the computation must take into account the 1PN relativistic correction in the quadrupole moment (and ADM mass). There is no simple way to express the new "enhancement" functions of eccentricity which appear at the 1PN order.
- One can check that the 1PN terms are a linear function of the symmetric mass ratio ν. Introduce two enhancement functions, denoted α and θ, s.t they are equal to one for circular orbits. The functions are defined as

$$\mathcal{F}_{\text{mass quad}} = \frac{32}{5}\,\nu^2\,x^{13/2}\left\{4\pi\,\varphi(e_t) + \pi\,x\left[-\frac{428}{21}\,\alpha(e_t) + \frac{178}{21}\nu\,\theta(e_t)\right]\right\}.$$

- Computed only by a numerical calculation.

Eccentricity enhancement factors

- At 1PN order we must use a specific definition for the eccentricity, and we adopted the eccentricity e_t. On the other hand, the variable $x = (m\omega)^{2/3}$ crucially incorporates the 1PN relativistic correction coming from the periastron advance $K = 1 + k$, through the definition $\omega = nK$.

Final expression of tail integrals

- Finally provide the complete results for the dimensionless enhancement factors and their numerical plots. Convenient for the final presentation to redefine in a minor way some of the enhancement functions by choosing

$$
\begin{aligned}
\psi(e) &= \frac{13696}{8191}\,\alpha(e) - \frac{16403}{24573}\,\beta(e) - \frac{112}{24573}\,\gamma(e),\\
\zeta(e) &= -\frac{1424}{4081}\,\theta(e) + \frac{16403}{12243}\,\beta(e) + \frac{16}{1749}\,\gamma(e),\\
\kappa(e) &= F(e) + \frac{59920}{116761}\chi(e).
\end{aligned}
$$

- Considering the 1.5PN and 2.5PN terms, composed of tails, and the 3PN terms, composed of tails of tails and tail squared, the total hereditary contribution to the *average* of the energy flux, normalized to the Newtonian value for circular orbits finally reads

Final expression of tail integrals

$$\mathcal{F}^{\rm 3PN}_{\rm tail} = \frac{32}{5}\,\nu^2\,x^5\Big\{4\pi\,x^{3/2}\,\varphi(e_t)+\pi\,x^{5/2}\left[-\frac{8191}{672}\,\psi(e_t)-\frac{583}{24}\nu\,\zeta(e_t)\right]$$
$$+x^3\left[-\frac{116761}{3675}\,\kappa(e_t)+\left[\frac{16}{3}\pi^2-\frac{1712}{105}C-\frac{1712}{105}\ln\left(\frac{4\omega r_0}{c}\right)\right]F(e_t)\right.$$

- All the enhancement functions are defined in such a way that they reduce to one in the circular case, $e_t=0$, so that the circular-limit of the formula is immediately seen from inspection and seen to be in complete agreement with Blanchet (98), Blanchet, Iyer Joguet (02)
- There are four enhancement functions which probably do not admit any analytic closed-form expressions: these are $\varphi(e_t)$, $\psi(e_t)$, $\theta(e_t)$ and $\kappa(e_t)$.

Final expression of tail integrals

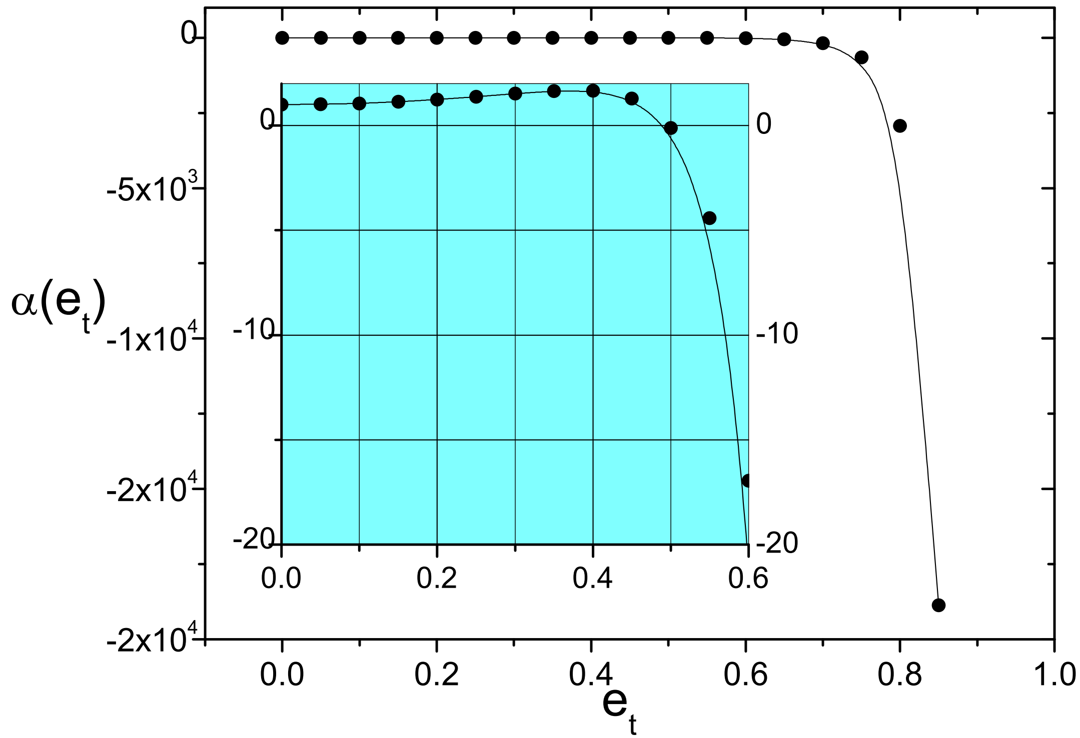
- Only $F(e_t)$ is known analytically, recall here its expression,

$$F(e_t) = \frac{1 + \frac{85}{6}e_t^2 + \frac{5171}{192}e_t^4 + \frac{1751}{192}e_t^6 + \frac{297}{1024}e_t^8}{(1-e_t^2)^{13/2}}.$$

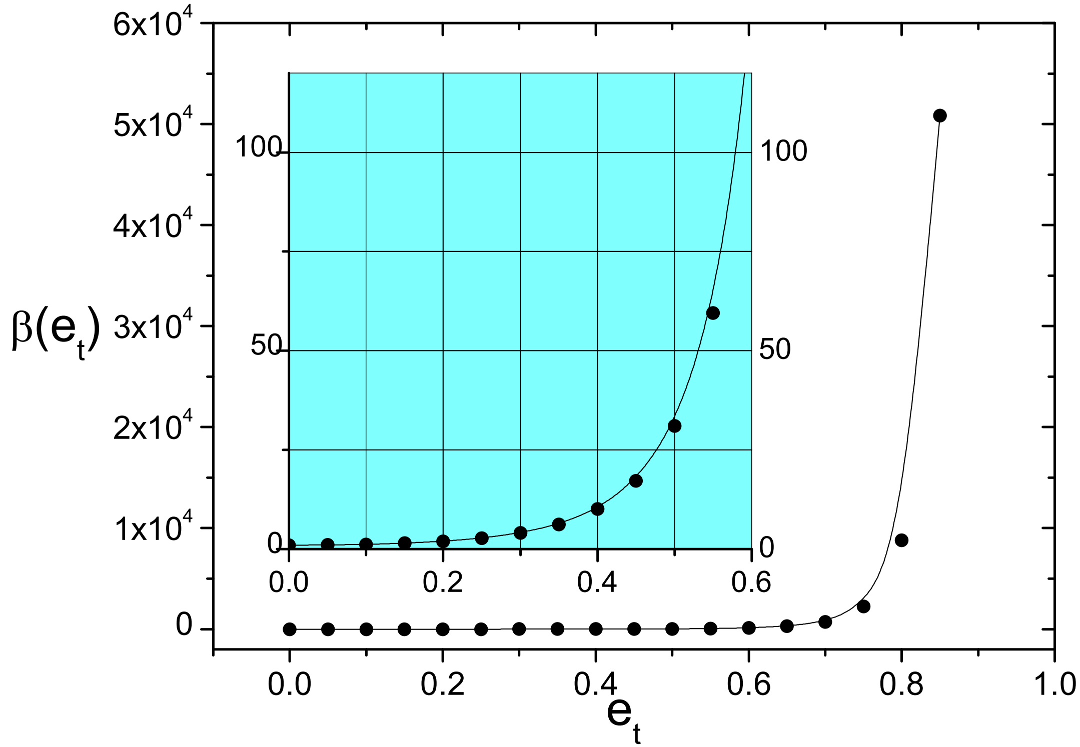
- The numerical plots of the four enhancement functions $\varphi(e_t)$, $\psi(e_t)$, $\theta(e_t)$ and $\kappa(e_t)$ displaying the plots of all these enhancement functions as functions of eccentricity e_t.

- Figures shows the variation of $\alpha(e_t)$, $\beta(e_t)$, $\gamma(e_t)$, $\theta(e_t)$, $\phi(e_t)$, $\psi(e_t)$, $\zeta(e_t)$ and $\kappa(e_t)$ with the eccentricity e_t. The inset is the zoom for the functions which look like straight horizontal lines in main graph. The dots are the numerical computation for the functions at $e_t =$0, 0.5, 1, 1.5,.... The solid lines are the fit to the numerical points. At the circular limit, $e_t = 0$, $\alpha(0) = \beta(0) = \gamma(0) = \theta(0) = \phi(0) = \psi(0) = \zeta(0) = \kappa(0) = 1$.

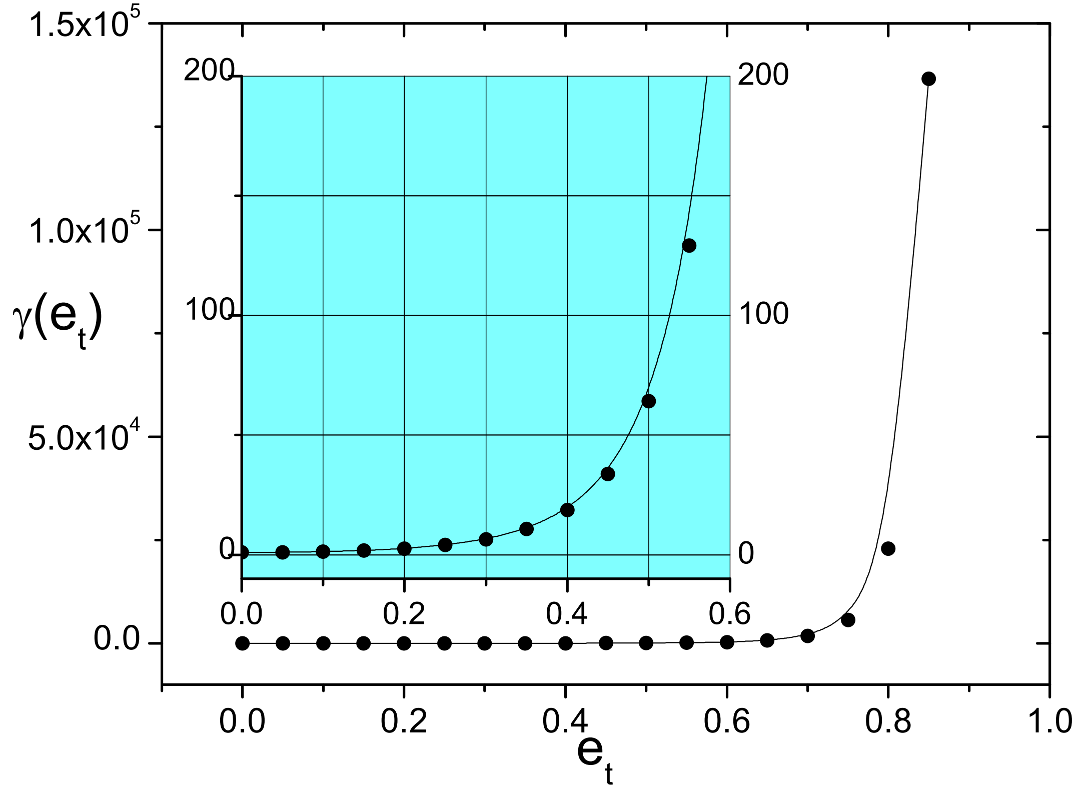
Enhancement Fn $\alpha(e_t)$



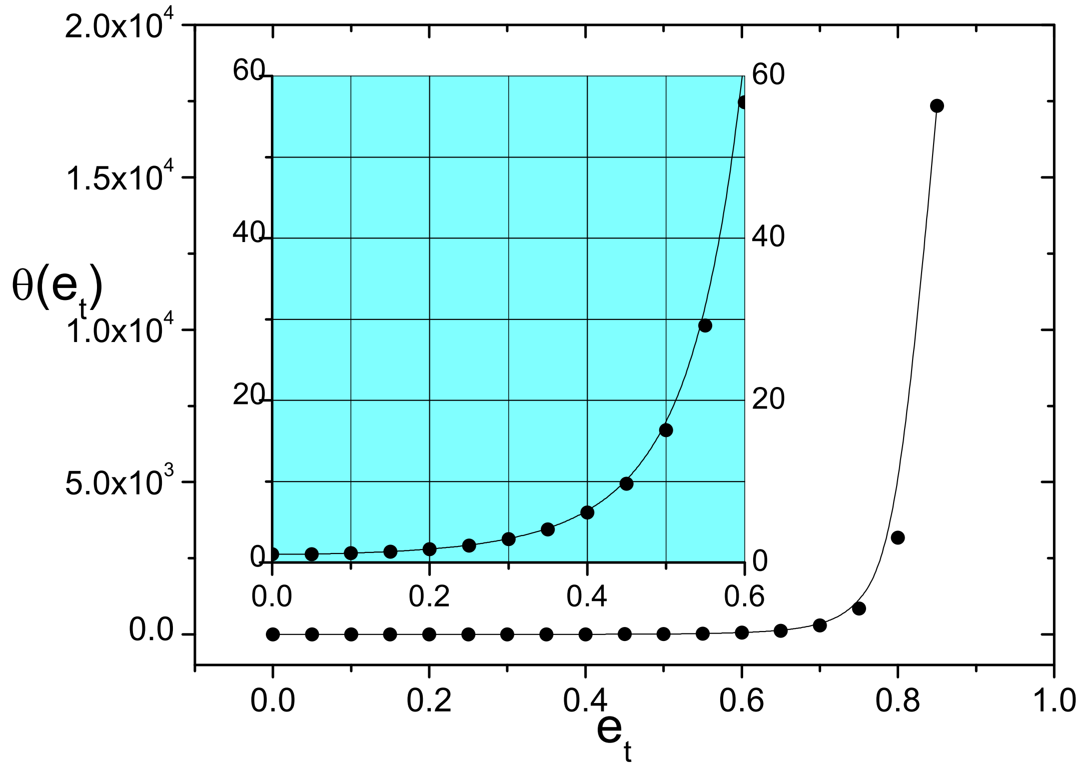
Enhancement Fn $\beta(e_t)$



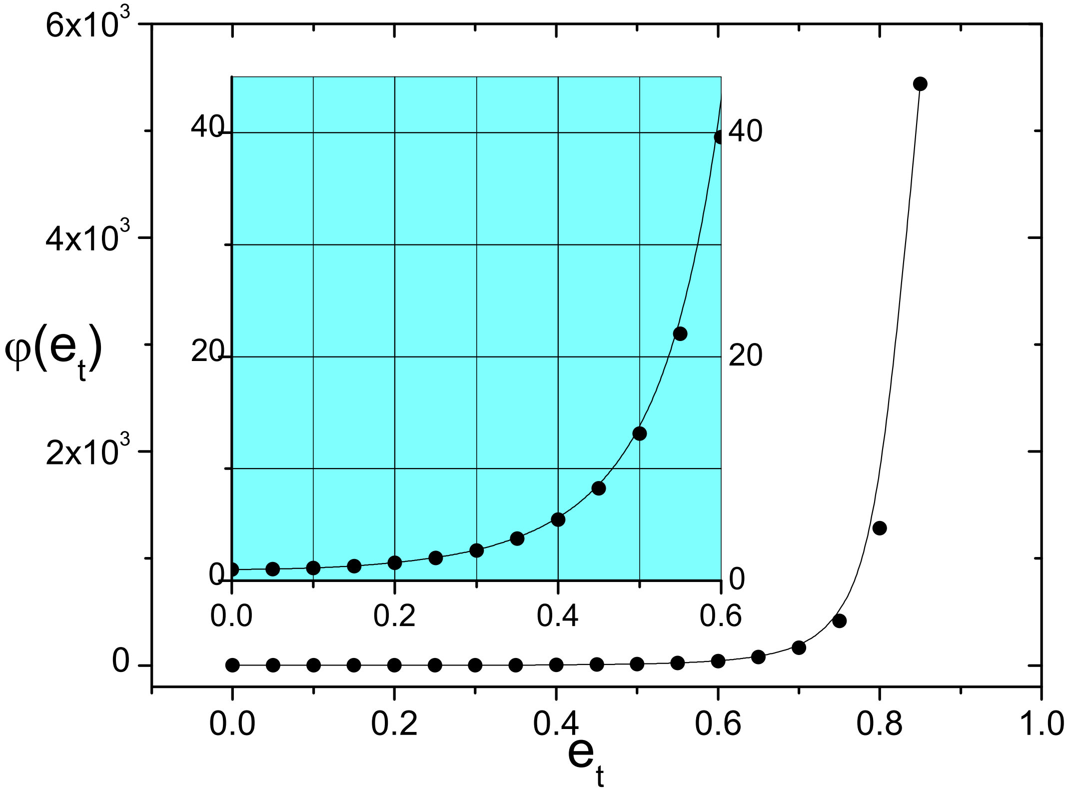
Enhancement Fn $\gamma(e_t)$



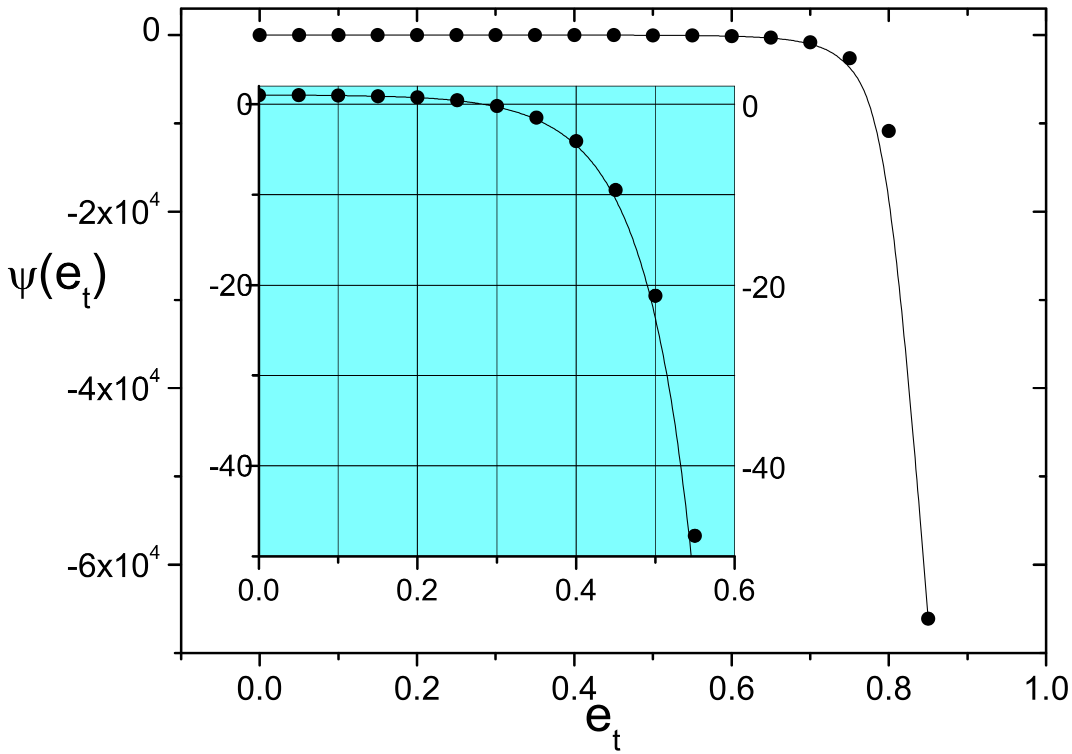
Enhancement Fn $\theta(e_t)$



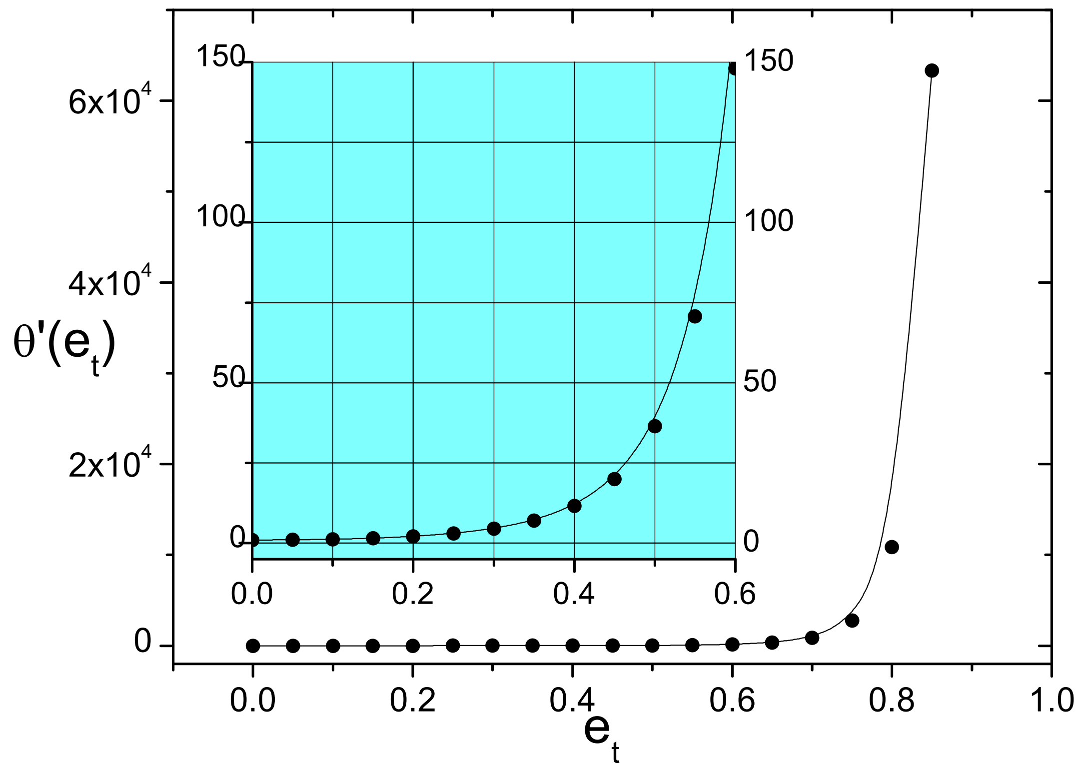
Enhancement Fn $\varphi(e_t)$



Enhancement Fn $\psi(e_t)$

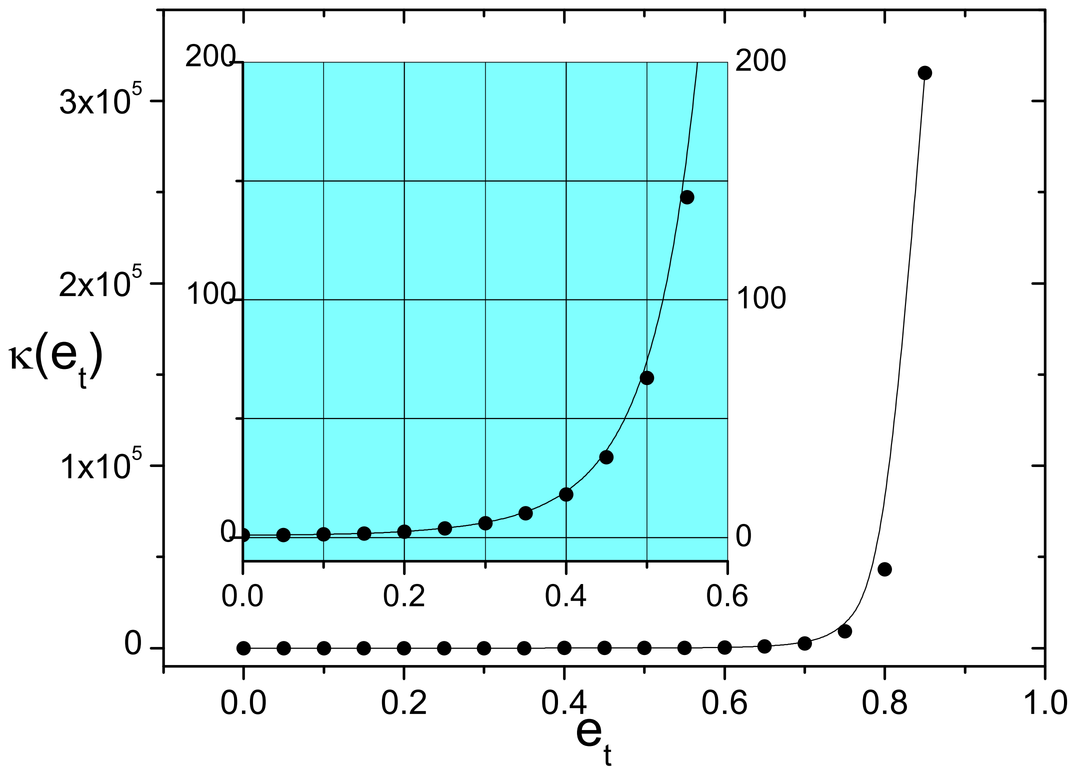


Enhancement Fn $\zeta(e_t)$



(Figure Label: $\theta' \equiv \zeta$)

Enhancement Fn $\kappa(e_t)$



3PN Instantaneous Terms

- The Instantaneous terms can be computed by a straightforward procedure in Standard harmonic coordinates in which our generation problem is set up
- These coordinates have some gauge dependent logarithms
- The computation of orbital averages requires the use of a GQKR representation at 3PN which is available in two gauges without the log terms - Modified Harmonic coords or ADM coordinates
- By a transformation of coords one can rewrite our fluxes in either of the coordinates and then average them by use of standard integrals involving Legendre functions

3PN generalised Quasi-Keplerian reprn

$$
\begin{aligned}
r &= a_r\,(1 - e_r\,\cos u)\ ,\\
l \equiv n\,(t - t_0) &= u - e_t\,\sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)(V - u)\\
&+ \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right)\sin V + \frac{i_{6t}}{c^6}\,\sin 2\,V + \frac{h_{6t}}{c^6}\,\sin 3\,V\ ,\\
\frac{2\,\pi}{\Phi}\,(\phi - \phi_0) &= V + \left(\frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6}\right)\sin 2V + \left(\frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6}\right)\sin 3V\\
&+ \frac{i_{6\phi}}{c^6}\,\sin 4V + \frac{h_{6\phi}}{c^6}\,\sin 5V\ ,\\
\text{where,}\ \ V &= 2\arctan\left[\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2}\tan\frac{u}{2}\right].
\end{aligned}
$$

3PN parametrization of the orbital motion of the binary was constructed by Memmeshiemer, Gopakumar and Schäfer (2004) in both ADM and modified harmonic coordinates.

Orbit Averaged Energy Flux - ADM (x, e_t)

$$x = \left(\frac{Gmn(1+k)}{c^3}\right)^{2/3}$$

$$\begin{aligned} < \dot{\mathcal{E}} >_{\rm ADM} &= \frac{32\nu^2 x^5}{5}\frac{1}{(1-e_t^2)^{7/2}}\Big(< \dot{\mathcal{E}}_{\rm N} >_{\rm ADM} + x < \dot{\mathcal{E}}_{\rm 1PN} >_{\rm ADM} \\ &\quad + x^2 < \dot{\mathcal{E}}_{\rm 2PN} >_{\rm ADM} + x^3 < \dot{\mathcal{E}}_{\rm 3PN} >_{\rm ADM}\Big)\,. \end{aligned}$$

$$\begin{aligned} < \dot{\mathcal{E}}_N >_{\rm ADM} &= 1 + e_t^2\frac{73}{24} + e_t^4\frac{37}{96} \\ < \dot{\mathcal{E}}_{\rm 1PN} >_{\rm ADM} &= \frac{1}{(1-e_t^2)}\Bigg\{\Bigg(\left(-\frac{1247}{336} - \frac{35}{12}\nu\right) + e_t^2\left(\frac{10475}{672} - \frac{1081}{36}\nu\right) \\ &\quad + e_t^4\left(\frac{10043}{384} - \frac{311}{12}\nu\right) + e_t^6\left(\frac{2179}{1792} - \frac{851}{576}\nu\right)\Bigg)\Bigg\} \end{aligned}$$

Orbit Averaged Energy Flux - ADM

$$
\begin{aligned}
< \dot{\mathcal{E}}_{\rm 2PN} >_{\rm ADM} \quad = \quad & \frac{1}{(1-e_t^2)^2} \\
& \Big\{ -\frac{203471}{9072} + \frac{12799}{504}\nu + \frac{65}{18}\nu^2 + e_t^2 \Big(-\frac{3866543}{18144} + \frac{4691}{2016}\nu + \frac{5935}{54} \\
& + e_t^4 \left(-\frac{369751}{24192} - \frac{3039083}{8064}\nu + \frac{247805}{864}\nu^2 \right) \\
& + e_t^6 \left(\frac{1302443}{16128} - \frac{215077}{1344}\nu + \frac{185305}{1728}\nu^2 \right) \\
& + e_t^8 \left(\frac{86567}{64512} - \frac{9769}{4608}\nu + \frac{21275}{6912}\nu^2 \right) \\
& + \sqrt{1-e_t^2} \left[\frac{35}{2} - 7\nu + e_t^2 \left(\frac{6425}{48} - \frac{1285}{24}\nu \right) \right. \\
& \left. + e_t^4 \left(\frac{5065}{64} - \frac{1013}{32}\nu \right) + e_t^6 \left(\frac{185}{96} - \frac{37}{48}\nu \right) \right] \Big\}
\end{aligned}
$$

Orbit Averaged Energy Flux - ADM

$$
\begin{aligned}
<\dot{\mathcal{E}}_{3\mathrm{PN}}>_{\mathrm{ADM}} = & \frac{1}{\left(1-e_t^2\right)^3} \\
& \left\{ \frac{1266161801}{9979200} + \left[\frac{8009293}{54432} - \frac{41}{64}\pi^2\right]\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3 \right. \\
& + e_t^2\left(\frac{27685797767}{19958400} + \left[\frac{249108317}{108864} + \frac{31255}{1536}\pi^2\right]\nu + \frac{133487}{6048}\nu^2 - \frac{53696}{243}\nu^3\right) \\
& + e_t^4\left(\frac{5135886353}{3326400} + \left[\frac{473750339}{108864} - \frac{7459}{1024}\pi^2\right]\nu + \frac{1305967}{576}\nu^2 - \frac{10816087}{7776}\nu^3\right) \\
& + e_t^6\left(\frac{352339259}{492800} + \left[-\frac{8775247}{145152} - \frac{78285}{4096}\pi^2\right]\nu + \frac{34228207}{12096}\nu^2 - \frac{983251}{648}\nu^3\right) \\
& + e_t^8\left(\frac{21840664301}{141926400} + \left[-\frac{36646949}{129024} - \frac{4059}{4096}\pi^2\right]\nu + \frac{86104369}{193536}\nu^2 - \frac{4586539}{15552}\nu^3\right) \\
& + e_t^{10}\left(-\frac{8977637}{11354112} + \frac{9287}{48384}\nu + \frac{8977}{55296}\nu^2 - \frac{567617}{124416}\nu^3\right) \\
& + \sqrt{1-e_t^2}\left[\left(-\frac{165761}{1008} + \frac{287}{192}\pi^2\right)\nu + e_t^2\left(-\frac{14935421}{6048} + \frac{52685}{4608}\pi^2\right)\nu \right. \\
& + e_t^4\left(-\frac{31082483}{8064} + \frac{41533}{6144}\pi^2\right)\nu + e_t^6\left(-\frac{40922933}{48384} + \frac{1517}{9216}\pi^2\right)\nu \\
& \left. + e_t^8\left(-\frac{1073}{288}\nu\right)\right] + \left(\frac{1712}{105} + \frac{14552}{63}e_t^2 + \frac{553297}{1260}e_t^4 + \frac{187357}{1260}e_t^6 \right. \\
& \left.\left. + \frac{10593}{2240}e_t^8\right)\ln\left[\left(\frac{c^2\, r_0}{Gm}x\right)\frac{1+\sqrt{1-e_t^2}}{2\left(1-e_t^2\right)}\right]\right\}
\end{aligned}
$$

Comments

- No 2.5PN term in the energy flux after averaging.
- Circular orbit limit as expected is in agreement with BIJ
- Newtonian and 1PN orders have the same form in Mhar coords and ADM coordinates because two coordinates differ starting only at 2PN.
- e_t in the above expression represents e_t^{ADM}, the time eccentricity in ADM coordinates.
- Checks involving cancellation of gauge dependent logs and logs related to regularisation at infinity involving the instantaneous terms and tail terms
- Provide gauge invariant expressions of the flux as suggested in MGS in terms of x and $k' = (\Phi - 2\pi)/6\pi = (K-1)/3 = k/3$

Comments

- Extends the circular orbit results at 2.5PN (Blanchet, 1990) and 3PN (Blanchet, Iyer, Joguet, 2002) to the elliptical orbit case. (involve both instantaneous and hereditary terms).
- Extends earlier works on instantaneous contributions for binaries moving in elliptical orbits at 1PN Blanchet Schäfer 89,Junker Schäfer 92) and 2PN (Gopakumar Iyer 97) to 3PN order.
- Extends hereditary contributions at 1.5PN by (Blanchet Schäfer 93) to 2.5PN order and 3PN.
- 3PN hereditary contributions comprise the *tail(tail)* and $tail^2$ and are extensions of (Blanchet 98) for circular orbits to the elliptical case.

Angular Momentum Flux

- For non-circular orbits, in addition to the conserved energy and gravitational wave energy flux, the angular momentum flux needs to be known to determine the phasing of eccentric binaries. A knowledge of the angular momentum flux of the system averaged over an orbit is mandatory to calculate the evolution of the orbital elements of non-circular, in particular, elliptic orbits under GW radiation reaction.
- We compute the angular momentum flux of inspiralling compact binaries moving in non-circular orbits up to 3PN order generalising earlier work at Newtonian order by Peters (1964), at 1PN order by Junker and Schäfer (Junker Schäfer 1992), 1.5PN (tails and spin-orbit) by Schaf́er and Rieth (1997) and at 2PN order by Gopakumar and Iyer (1997). Unlike at earlier post-Newtonian orders, the 3PN contribution to angular momentum flux comes not only from *instantaneous* terms but also *hereditary* contributions.

Far Zone Angular Momentum Flux

$$\left(\frac{d\mathcal{J}_i}{dt}\right) = \frac{G}{c^5}\epsilon_{ipq}\left\{\frac{2}{5}U_{pj}U^{(1)}_{qj}\right.$$
$$+\frac{1}{c^2}\left[\frac{1}{63}U_{pjk}U^{(1)}_{qjk}+\frac{32}{45}V_{pj}V^{(1)}_{qj}\right]+\frac{1}{c^4}\left[\frac{1}{2268}U_{pjkl}U^{(1)}_{qjkl}+\frac{1}{28}V_{pjk}V^{(1)}_{qjk}\right]$$
$$\left.+\frac{1}{c^6}\left[\frac{1}{118800}U_{pjklm}U^{(1)}_{qjklm}+\frac{16}{14175}V_{pjkl}V^{(1)}_{qjkl}\right]+\mathcal{O}(8)\right\}.$$

- Using the MPM formalism, the radiative moments can be re-expressed in terms of the source moments to an accuracy sufficient for the computation of the angular momentum flux up to 3PN.
- For the AM flux to be complete up to 3PN approximation, one must compute the mass type radiative quadrupole U_{ij} to 3PN accuracy, mass octupole U_{ijk} and current quadrupole V_{ij} to 2PN accuracy, mass hexadecupole U_{ijkm} and current octupole V_{ijk} to 1PN accuracy and finally U_{ijkmn} and V_{ijkm} to Newtonian accuracy.

Orbital Averaged AMF - ADM

$$\zeta = \frac{G\,m\,n}{c^3}$$

$$\begin{aligned}\left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm ADM}_{\rm inst} &= \frac{4}{5}\,c^2\,m\,\zeta^{7/3}\,\nu^2\,\frac{1}{(1-{e_t}^2)^{7/2}}\left[\left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm Newt}+\left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm 1PN}+\left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm 2PN}\right.\\ &\left.+\left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm 2.5PN}+\left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm 3PN}\right],\end{aligned}$$

where $\zeta = \frac{G\,m\,n}{c^3}$ and the individual terms read as:

$$\begin{aligned}\left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm Newt} &= \frac{8+7e_t^2}{(1-e_t^2)^2}\,,\\ \left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm 1PN} &= \zeta^{2/3}\frac{1}{(1-e_t)^3}\left\{\frac{1105}{42}-\frac{70\nu}{3}+e_t^2\left[\frac{5077}{42}-\frac{335\nu}{3}\right]\right.\\ &\left.+e_t^4\left[\frac{8399}{336}-\frac{275\nu}{12}\right]\right\},\end{aligned}$$

Orbital Averaged AMF - ADM

$$
\begin{aligned}
\langle \frac{d\mathcal{J}}{dt} \rangle^{\mathrm{2PN}} \quad = \quad & \zeta^{4/3}\, \frac{1}{(1-e_t^2)^4} \Bigg\{ \left[\frac{7238}{81} - \frac{10175\,\nu}{63} + \frac{260\,\nu^2}{9} \right] \\
& + e_t^2 \left[\frac{376751}{756} - \frac{37047\,\nu}{28} + \frac{1546\,\nu^2}{3} \right] \\
& + e_t^4 \left[\frac{377845}{756} - \frac{168863\,\nu}{168} + 569\,\nu^2 \right] \\
& + e_t^6 \left[\frac{30505}{2016} - \frac{2201\,\nu}{56} + \frac{1519\,\nu^2}{36} \right] \\
& + \sqrt{1-e_t^2} \left[80 - 32\,\nu + e_t^2 (335 - 134\,\nu) + e_t^4 (35 - 14\,\nu) \right] \Bigg\} \,,
\end{aligned}
$$

Orbital Averaged AMF - ADM

$$
\begin{aligned}
\langle \frac{d\mathcal{J}}{dt} \rangle^{3\mathrm{PN}} \quad = \quad & \zeta^2 \frac{1}{\left(1 - e_t^2\right)^5} \Bigg\{ \left[\frac{265845199}{138600} - \frac{20318135\,\nu}{6804} + \frac{287\pi^2\,\nu}{4} + \frac{187249\,\nu^2}{378} - \frac{1550\,\nu^3}{81} \right] \\
& + e_t^2 \left[\frac{1476919051}{178200} - \frac{82215823\,\nu}{6804} + \frac{5171\pi^2\,\nu}{32} + \frac{387467\,\nu^2}{54} - \frac{96973\,\nu^3}{81} \right] \\
& + e_t^4 \left[\frac{669008149}{103950} - \frac{206700631\,\nu}{18144} - \frac{2799\pi^2\,\nu}{256} + \frac{13341787\,\nu^2}{1008} - \frac{438907\,\nu^3}{108} \right] \\
& + e_t^6 \left[\frac{114553217}{123200} - \frac{39280525\,\nu}{18144} - \frac{615\pi^2\,\nu}{128} + \frac{1092025\,\nu^2}{336} - \frac{283205\,\nu^3}{162} \right] \\
& + e_t^8 \left[-\frac{10305073}{709632} + \frac{417923\,\nu}{12096} + \frac{95413\,\nu^2}{8064} - \frac{146671\,\nu^3}{2592} \right] \\
& + \left[-\frac{13696}{105} - \frac{98012 e_t^2}{105} - \frac{23326 e_t^4}{35} - \frac{2461 e_t^6}{70} \right] \log \left[\frac{2\left(1 - e_t^2\right) G\, m}{c^2 \left(\sqrt{1 - e_t^2} + 1\right) r_0\, \zeta^{2/3}} \right]
\end{aligned}
$$

Orbital Averaged AMF - ADM

$$
\begin{aligned}
&+\sqrt{1-e_t^2}\left[\frac{351577}{630}-\frac{78139\,\nu}{63}+\frac{41\pi^2\,\nu}{6}+\frac{580\,\nu^2}{3}\right.\\
&+e_t^2\left[\frac{1723433}{315}-\frac{569398\,\nu}{63}+\frac{2747\pi^2\,\nu}{96}+1902\,\nu^2\right]\\
&+e_t^4\left[\frac{17557661}{5040}-\frac{2444195\,\nu}{504}+\frac{287\pi^2\,\nu}{96}+\frac{2703\,\nu^2}{2}\right]\\
&\left.\left.+e_t^6\left[70-\frac{203\,\nu}{3}+\frac{77\,\nu^2}{3}\right]\right]\right\}.
\end{aligned}
$$

Checks

- Circular orbit limit ($e_t = 0$) As an algebraic check, we take the circular orbit limit of the orbital average of angular momentum flux and the energy flux in ADM coordinates expressed in terms of ζ and e_t. For circular orbit binaries the angular momentum flux and the energy flux must be simply related as

$$\frac{d\mathcal{E}}{dt} = \omega \, \frac{d\mathcal{J}}{dt}$$

in any coordinate system. Here $\frac{d\mathcal{J}}{dt}$ is the magnitude of the angular momentum flux.

- The circular orbit limit of our calculation agrees with the above expression with ω and is given by

$$\begin{aligned} \omega &= \left(\frac{c^3\,\zeta}{G\,m}\right)\left\{1 + 3\,\zeta^{2/3} + \zeta^{4/3}\left[\frac{39}{2} - 7\nu\right]\right. \\ &\left. +\zeta^2\left[\frac{315}{2} + \frac{1}{32}\left(-6536 + 123\pi^2\right)\nu + 7\nu^2\right]\right\}, \end{aligned}$$

where $\zeta = \frac{G\,m\,n}{c^3}$.

Evoln of orbital elements under GRR

- Most important application of the 3PN angular momentum flux obtained here and the energy flux obtained is to calculate how the orbital elements of the binary evolve with time under GRR. By 3PN evolution of orbital elements under GRR we mean its evolution under 5.5PN terms beyond leading newtonian order in the EOM.
- We compute the rate of change of n, e_t and a_r averaged over an orbit, due to GRR.
- We start with the 3PN accurate expressions for n and e_t in terms of the 3PN conserved energy (E) and angular momentum (J). Differentiating them w.r.t time and using *heuristic balance equations* for energy and angular momentum up to 3PN order, we compute the rate of change of the orbital elements.
- Extends the earlier analyses at Newtonian order by Peters (64), 1PN computation of Blanchet Schäfer 89,Junker Schäfer 92 and at 2PN order by Gopakumar Iyer 97,Damour Gopakumar Iyer 04. The 1.5PN hereditary effects also have been accounted in the orbital element evolution in Blanchet Schäfer 93, Rieth Schäfer 97.

Evoln of orbital elements under GRR

- 3PN accurate expressions for the mean motion n, eccentricity e_t and semi-major axis a_r read are listed. Let us use the example of n to outline the procedure adopted for the computation of orbital elements in more detail. The expression for n is symbolically written as

$$n = n(E, J).$$

Differentiating with respect to t one obtains

$$\frac{dn}{dt} = \gamma_1(e_t, \zeta, \nu)\,\frac{dE}{dt} + \gamma_2(e_t, \zeta, \nu)\,\frac{d|\mathbf{J}|}{dt},$$

where γ_1 and γ_2 are PN expansions in powers of ζ. Now we use the balance equations,

$$\begin{aligned}\frac{dE}{dt} &= -\frac{d\mathcal{E}}{dt},\\ \frac{d|\mathbf{J}|}{dt} &= -\frac{d\mathcal{J}}{dt}.\end{aligned}$$

Evoln of orbital element n under GRR

- Replace the time derivatives of the conserved energy and angular momentum (on the right side of the expression for $\frac{dn}{dt}$) with the energy and angular momentum fluxes and compute the final expression for the orbital average by using the orbital averages of the energy and angular momentum fluxes up to 3PN. It may be noted that, the angular momentum flux is needed only up to 1PN accuracy for the computation of $\langle \frac{dn}{dt} \rangle$ where as the energy flux is needed up to 3PN. The structure of the evolution equations is similar for the other orbital elements also and the same procedure can be employed. The final expression for the 3PN evolution of n reads

$$\langle \frac{dn}{dt} \rangle^{\mathrm{ADM}}_{\mathrm{inst}} = \frac{c^6}{G^2\, m^2}\, \zeta^{11/3} \left[\langle \frac{dn}{dt} \rangle_{\mathrm{Newt}} + \langle \frac{dn}{dt} \rangle_{\mathrm{1PN}} + \langle \frac{dn}{dt} \rangle_{\mathrm{2PN}} + \langle \frac{dn}{dt} \rangle_{\mathrm{3PN}} \right]$$

Evoln of orbital element n under GRR

$$
\begin{aligned}
\langle \frac{dn}{dt} \rangle_{\rm Newt} &= \frac{1}{\left(1-e_t^2\right)^{7/2}} \left\{ \frac{96}{5} + \frac{292 e_t^2}{5} + \frac{37 e_t^4}{5} \right\} \,, \\
\langle \frac{dn}{dt} \rangle_{\rm 1PN} &= \frac{\zeta^{2/3}}{\left(1-e_t^2\right)^{9/2}} \left\{ \frac{2546}{35} - \frac{264\,\nu}{5} + e_t^2 \left[\frac{5497}{7} - 570\,\nu \right] \right. \\
&+ \left. e_t^4 \left[\frac{14073}{20} - \frac{5061\,\nu}{10} \right] + e_t^6 \left[\frac{11717}{280} - \frac{148\,\nu}{5} \right] \right\} \,, \\
\langle \frac{dn}{dt} \rangle_{\rm 2PN} &= \frac{\zeta^{4/3}}{\left(1-e_t^2\right)^{11/2}} \left\{ \frac{393527}{945} + e_t^2 \left[\frac{4098457}{945} - \frac{108047\nu}{15} + \frac{182387\nu^2}{90} \right] \right. \\
&+ e_t^4 \left[\frac{1678961}{180} - \frac{2098263\nu}{140} + \frac{396443\nu^2}{72} \right] + e_t^6 \left[\frac{1249229}{336} - \frac{76689\nu}{16} + \frac{19294}{90} \right. \\
&+ \sqrt{1-e_t^2} \left[48 - \frac{47491\nu}{105} + \frac{944\nu^2}{15} + e_t^2 \left[2134 - \frac{4268\nu}{5} \right] + e_t^4 \left[2193 - \frac{4386}{5} \right. \right. \\
&+ \left. e_t^6 \left[\frac{175}{2} - 35\nu \right] - \frac{96\nu}{5} \right] + e_t^8 \left[\frac{391457}{3360} - \frac{6037\nu}{56} + \frac{2923\nu^2}{45} \right] \left. \right\} \,,
\end{aligned}
$$

Evoln of orbital element n under GRR

$$
\begin{aligned}
\langle \frac{dn}{dt} \rangle_{3\mathrm{PN}} &= \frac{\zeta^2}{\left(1-e_t^2\right)^{13/2}} \Bigg\{ \Bigg[\frac{6687854333}{1039500} - \frac{113898769\,\nu}{11340} + \frac{2337\pi^2\,\nu}{10} + \frac{564197\,\nu^2}{420} - \frac{1121\,\nu^3}{27} \\
&+ e_t^2 \Bigg[\frac{132891898933}{2079000} - \frac{1993945913\,\nu}{22680} + \frac{19207\pi^2\,\nu}{16} + \frac{5552087\,\nu^2}{168} - \frac{1287385\,\nu^3}{324} \Bigg] \\
&+ e_t^4 \Bigg[\frac{151872497839}{1188000} - \frac{2340827549\,\nu}{12960} + \frac{22723\pi^2\,\nu}{160} + \frac{28833055\,\nu^2}{224} - \frac{33769597\,\nu^3}{1296} \Bigg] \\
&+ e_t^6 \Bigg[\frac{63380900591}{792000} - \frac{2509038229\,\nu}{20160} - \frac{43937\pi^2\,\nu}{128} + \frac{236136203\,\nu^2}{2240} - \frac{3200965\,\nu^3}{108} \Bigg] \\
&+ e_t^8 \Bigg[\frac{93247526201}{7392000} - \frac{1814291\,\nu}{96} - \frac{12177\pi^2\,\nu}{640} + \frac{3251909\,\nu^2}{210} - \frac{982645\,\nu^3}{162} \Bigg] +
\end{aligned}
$$

Evoln of orbital element n under GRR

$$
\begin{aligned}
+ \quad & e_t^{10}\left[\frac{33332681}{197120}-\frac{1874543\,\nu}{10080}+\frac{109733\,\nu^2}{840}-\frac{8288\,\nu^3}{81}\right]\Bigg]\\
+ \quad & \sqrt{1-e_t^2}\,\Bigg[\Bigg[-\frac{669319}{1125}-\frac{3670\,\nu}{21}-\frac{41\pi^2\,\nu}{10}+\frac{632\,\nu^2}{5}\\
+ \quad & e_t^2\left[\frac{11326954}{375}-\frac{14778121\,\nu}{315}+\frac{45961\pi^2\,\nu}{240}+\frac{125278\,\nu^2}{15}\right]\\
+ \quad & e_t^4\left[\frac{1534643951}{21000}-\frac{5720941\,\nu}{60}+\frac{6191\pi^2\,\nu}{32}+\frac{317273\,\nu^2}{15}\right]\\
+ \quad & e_t^6\left[\frac{775558207}{31500}-\frac{35318351\,\nu}{1260}+\frac{287\pi^2\,\nu}{960}+\frac{232177\,\nu^2}{30}\right]\\
+ \quad & e_t^8\left[\frac{56403}{112}-\frac{427733\,\nu}{840}+\frac{4739\,\nu^2}{30}\right]\Bigg]\Bigg]\\
+ \quad & \left.\frac{107\left[3072+43520e_t^2+82736e_t^4+28016e_t^6+891e_t^8\right]\ln\left[X\right]}{1050\left[1-e_t^2\right]^{13/2}}\right\},
\end{aligned}
$$

$$
X=\left[\frac{c^2\left(\sqrt{1-e_t^2}+1\right)\,r_0\,\zeta^{2/3}}{2\left(1-e_t^2\right)\,G\,m}\right].
$$

Evoln of orbital elements under GRR

- The three expressions obtained here are the 3PN generalizations of the expressions given in Peters which are at the lowest quadrupolar order. They could be used to provide 3PN extensions of $n(e)$ and $a(e)$ relations in the future.
- The above results have to be supplemented by the computation of *hereditary* terms at 2.5PN and 3PN for completion. These hereditary terms include the tails at 2.5PN and tail of tails and tail-square terms at 3PN.
- Formally one can analytically solve the coupled evolution system by successive approximations, reducing it to simple quadratures. Eg, at the leading order $\mathcal{O}(c^{-5})$ one can first eliminate t by dividing $d\bar{n}/dt$ by $d\bar{e}_t/dt$, thereby obtaining an equation of the form $d\ln\bar{n} = f_0(\bar{e}_t)d\bar{e}_t$. Integration of this equation yields

$$\bar{n}(\bar{e}_t) = n_i \, \frac{e_i^{18/19}\,(304 + 121\,e_i^2)^{1305/2299}}{(1-e_i^2)^{3/2}} \, \frac{(1-e_t^2)^{3/2}}{e_t^{18/19}\,(304 + 121\,e_t^2)^{1305/2299}}\,,$$

e_i is the value of e_t when $n = n_i$. First obtained by Peters 64.

Beyond Orbital Averages

- GW obsvns of ICB, are analogous to the high precision Radio wave obsvns of binary pulsars.
- GW obsvns demand accurate 'phasing', i.e. an accurate mathematical modeling of the continuous time evolution of the gravitational waveform.
- GW emitted from inspiralling *circular* orbits, contain only two different time scales: orbital motion and radiation reaction
- *Inspiralling eccentric* orbits involve *three different time scales*: orbital period, periastron precession and radiation-reaction time scales.
- By using an improved 'method of variation of constants', one can combine these three time scales, without making the usual approximation of treating the radiative time scale as an adiabatic process. Relies on techniques from (Damour 83, 85) to implement PN 'phasing' for elliptical orbits.

Beyond Orbital Averages

- Going beyond the average evolution of the orbit under Grav Radn reaction the method allows one to deal with both a 'slow' (radiation-reaction time-scale) secular drift and 'fast' (orbital time-scale) periodic oscillations.
- Method implemented at the 2.5PN (Damour, Iyer, Gopakumar) and 3.5PN (Königsdörffer, Gopakumar)
- Results compute new 'post-adiabatic' short period contributions to the orbital phasing, or equivalently, new short-period contributions to GW polarizations, $h_{+,\times}$, to be explicitly added to PN expn for $h_{+,\times}$, if one treats radiative effects on the orbital phasing in the usual adiabatic approximation.
- Should be of importance both for the LIGO/VIRGO/GEO network of ground based interferometric GW detectors and for space-based interferometer LISA.

Refer to

Phasing of Gravitational waves fluxes from inspiralling eccentric binaries 2.5PN/3.5PN

T. Damour, A. Gopakumar and B. R. Iyer

Phys.Rev. D70 (2004) 064028

C. Koenigsdoerffer, A. Gopakumar

Phys.Rev. D73 (2006) 124012

1PN Quasi-Keplerian Reprn

The explicit dependence of the orbital elements in terms of the 1PN conserved orbital energy E and angular momentum J is given in (Damour Deruelle 85)

$$
\begin{aligned}
a_r &= \frac{1}{Gm}\frac{1}{(-2\,E)}\left\{1+\frac{(-2\,E)}{4\,c^2}\,(-7+\nu)\right\},\\
{e_r}^2 &= 1+2\,E\,h^2+\frac{(-2\,E)}{4\,c^2}\left\{24-4\,\nu+5\,(-3+\nu)\,(-2\,E\,h^2)\right\},\\
n &= (-2\,E)^{3/2}\left\{1+\frac{(-2\,E)}{8\,c^2}\,(-15+\nu)\right\},\\
{e_t}^2 &= 1+2\,E\,h^2+\frac{(-2\,E)}{4\,c^2}\left\{-8+8\,\nu-(-17+7\,\nu)\,(-2\,E\,h^2)\right\},\\
{e_\phi}^2 &= 1+2\,E\,h^2+\frac{(-2\,E)}{4\,c^2}\left\{24+(-15+\nu)\,(-2\,E\,h^2)\right\}.
\end{aligned}
$$