Higher-order spin effects in the dynamics of compact binaries

Guillaume Faye[a] Luc Blanchet[a] Alessandra Buonanno[b,c,a]

[a] Institut d'Astrophysique de Paris (IAP), UMR 7095 CNRS
Université Pierre & Marie Curie
[b] Department of Physics, University of Maryland
[c] AstroParticule et Cosmologie (APC)

14th November 2006

Motivation

- black-hole binary systems = important source of GW detectable by LISA (or Virgo)

In particular

- in-spiral and coalescence of massive black holes
- extreme mass-ratio in-spiraling (EMRI) events
- in-spiral of stellar black holes ($\rightarrow$ Virgo – LIGO)

- matter accretion of black holes in their lives
  $\hookrightarrow$ increasing of the spin i.e. intrinsic angular momentum

$\Rightarrow$

model for the in-spiral of a spinning black-hole compact binary needed: continuation of a work by Tagoshi et al. and Owen et al.

[PRD **63** (2001), PRD **57** (1998)]

The post-Newtonian scheme

- post-Newtonian approximation for a in-spiraling binary of typical mass m and typical separation $r_{12} \sim L$
  - weak fields and velocities: $\dfrac{v^2}{c^2} \sim \dfrac{Gm}{Lc^2} \ll 1$
  - weak stress: $\sigma = \dfrac{T^{00} + T^{ii}}{c^2}$, $\sigma_i = \dfrac{T^{0i}}{c}$, $\sigma_{ij} = T^{ij}$

  $\Bigg\}$ Newt. quant.

- **post-Newtonian expansion**

  perturbative expansion of the metric in powers of $1/c^2 \ll 1$ (for $G = m = L = 1$)
  Notation: *1PN = order* $1/c^2$

- domain of validity
  $L/\lambda \sim v/c \longrightarrow$ existence of a near zone $\mathcal{D}_{\text{near}}$ of typical size D with negligible GW-propagation effects
  $\Rightarrow r = |\mathbf{x}| \ll \lambda$ in $\mathcal{D}_{\text{near}}$

Application to the computation of the orbital phase

1. Specialization to binary in-spiraling systems
   - consequence of GW radiation a few 1000's of cycles before the coalescence
     - motion $\sim$ circular with orbital frequency ω
     - binding energy $E \searrow \Rightarrow \omega \nearrow$ and $r \searrow$ (Kepler law)
   - for $r \gtrsim 10m$ (say), PN approximation valid

adiabatic in-spiral

nearly circular orbits with $E \searrow$ adiabatically according to $dE/dt =$ - (GW flux $\mathcal{L}$) in the PN regime

2. resulting strategy in the computation of $\Phi_{\mathrm{GW}} = 2\phi_{\mathrm{orb.}} + \overset{\text{precessional correction}}{\delta\Phi_{\mathrm{GW}}}$

$$E(\omega),\ \mathcal{L}(\omega) \text{ PN scheme} \quad \Rightarrow \quad \frac{d\omega}{dt} = \frac{-\mathcal{L}}{dE/d\omega} \quad \Rightarrow \quad \phi_{\mathrm{orb.}}$$

A point-particle model

up to very high PN orders, in the absence of the spin

compact objects = point-particles + self-field elimination by dimensional regularization procedure

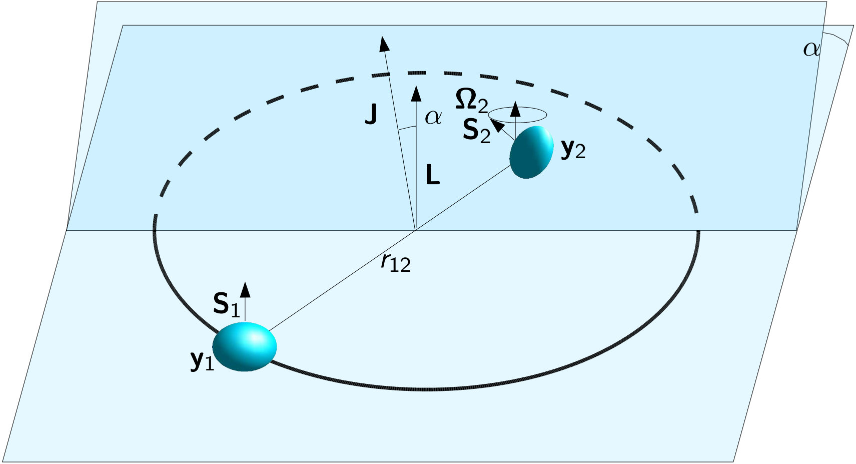
3 justifications

- general arguments showing that tidal effects appear at 5PN
- agreement of the equations of motion for a compact binary with those obtained in the extended-body approach by Itoh
- Brill-Lindquist initial solution for two black holes recovered by means of point-particles

up to 2PN, Hadamard regularization sufficient

- self-field singularities near body 1 at position $\mathbf{y}_1$ removed
- angular average over $\mathbf{n}_1 = (\mathbf{x} - \mathbf{y}_1)/|\mathbf{x} - \mathbf{y}_1|$ performed
- limit $r_1 = |\mathbf{x} - \mathbf{y}_1| \to 0$ taken

Notation


α
$\mathbf{J}$
α
Ω_2
$\mathbf{S}_2$
$\mathbf{y}_2$
$\mathbf{L}$
r_{12}
$\mathbf{S}_1$
$\mathbf{y}_1$

Spin modeling I

definition: spin tensor for a test particle

anti-symmetric 4-tensor parameterizing the dipole of the stress-energy tensor

$$T^{\mu\nu} = c\sum_A \left[\underbrace{u_A^{(\mu}}_{\text{part. 4-velocity}} \underbrace{p_A^{\nu)}}_{\text{part. momentum}} u_A^0 \frac{\delta(\mathbf{x}-\mathbf{y}_A)}{\sqrt{-\underbrace{g_A}_{\det g_{\mu\nu} \text{ at } \mathbf{y}_A}}} - \nabla_\rho \left(\underbrace{S^{\rho(\mu}}_{\text{part. spin tensor}} u_A^{\nu)} u_A^0 \frac{\delta(\mathbf{x}-\mathbf{y}_A)}{\sqrt{-g_A}} \right) \right]$$

p_A^μ and $S_A^{\mu\nu}$ chosen consistently so that $\dfrac{DS^{\mu\nu}}{d\tau} \equiv c(p_A^\mu u_A^\nu - p_A^\nu u_A^\mu)$

$\hookrightarrow$ still arbitrariness in the definition of $S_A^{\mu\nu}$

introduction of spin supplementary conditions (SSC)

$S_A^{\mu\nu} p_\nu^A = 0 \quad \Rightarrow \quad$ dual of $S_A^{\mu\nu}$ of the form $S_A^\mu p_A^\nu$

$S_\mu^A u_A^\mu = 0$

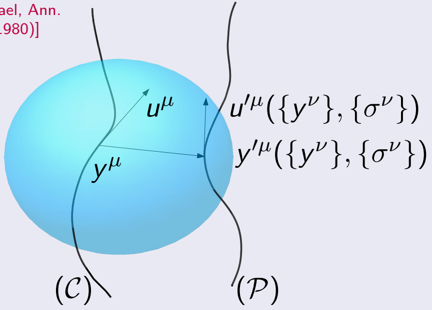
Spin modeling II

Consequences:

- $p_A^\mu p_\mu^A = -m_A^2 c^2$
- spin entirely encoded by a 3-vector $S_i^A \equiv g_{ij} S_A^j$

physical interpretation for extended bodies

[Bailey & Israel, Ann. Phys. **130** (1980)]



present model

$=$ test spinning particle model $+$ Hadamard reg.
$\hookrightarrow g_A \equiv$ reg. of g at $\mathbf{y}_A$

Spin modeling III

Bodies with extension $\ll$ inverse curvature: $T^{\mu\nu}$ action as a distribution simplified

$$\int d^4x T^{\mu\nu}(x)\phi_{\mu\nu}(x) \approx \int d^4x T^{\mu\nu}_{\text{skeleton}}(x)\phi_{\mu\nu}(x)$$

$$\text{with } T^{\mu\nu}_{\text{skeleton}} = \int d\tau \sum_{\ell=0}^{+\infty} \nabla_{\mu_1}\ldots\nabla_{\mu_\ell}\left[\mathcal{I}^{\mu\nu\mu_1\ldots\mu_\ell}\frac{\delta^4(x-y(\tau))}{\sqrt{-g}}\right]$$

constraints of conservation $\nabla_\nu T^{\mu\nu}_{\text{skeleton}} = 0$

relations between the momenta and the 4-velocity $u^\mu = dx^\mu/d\tau$

for $\ell \leq 1 \rightarrow T^{\mu\nu}_{\text{skeleton}} =$ stress-energy of a point-like object with spin

Spin scaling

1. Orders of magnitude
$S \sim |\mathbf{x}_{\text{object}} \times \mathbf{p}_{\text{spin}}| \lesssim mrv_{\text{spin}} = (Gm^2/c)[rc^2/(Gm)](v_{\text{spin}}/c)$

for a compact object $rc^2/(Gm) \sim 1$

- fast rot. $\Rightarrow v_{\text{spin}}/c \sim 1$ $\quad S \sim Gm^2/c = \mathcal{O}(1/c)$
- slow rot. $\Rightarrow v_{\text{spin}}/c \ll 1$ $\quad S \sim Gm^2 v_{\text{spin}}/c^2 = \mathcal{O}(1/c^2)$

Notation: $S_A^i = cS^i_{A\,(\text{real})}$

2. Dominant spin effects on the dynamics
acceleration SO: $\propto (G\partial^2 r_{12}^{-1})\epsilon_{\text{Levi Civita}} v_B S_A/c^3 \rightarrow 1.5\text{PN}$
acceleration SS: $\propto (Gn_{12}/r_{12}^2)S_1 S_2(n_{12}n_{12})/(m_A c^4) \rightarrow 2\text{PN}$

to be computed at 2.5PN

first corrections of SO interactions
SS terms ignored (already known at 2PN)

Newtonian metric in the near zone I

Perturpative calculation

in the near zone $\mathcal{D}_{\text{near}}$ metric searched under the form of a PN expansion $g_{\mu\nu} = \sum_{m > m_0(\mu\nu)} g^{(m)}_{\mu\nu}/c^m$

Main steps of the general algorithm

1. Starting point: Einstein equations in harmonic coordinates (with $h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu}$) flat metric

   inverse metric

   $$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}|g|\, T^{\mu\nu} + \Lambda^{\mu\nu}(h,h), \qquad \partial_\nu h^{\mu\nu} = 0$$

2. Iterative computation of the metric
   - solving of Einstein equations at order m with $\Box^{-1}_{\mathcal{R}} \overset{\text{3.5PN}}{=}$ ret. int.
   - replacement of h in $\Lambda^{\mu\nu}(h,h)$ at order $m+2 \longrightarrow \Lambda^{\mu\nu}_{(m+2)}(h,h)$
   - passing over to next order

Post-Newtonian metric in the near zone II

$$g_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left[\hat{X} + V_i V_i + \frac{V^3}{6}\right] + \mathcal{O}\left(\frac{1}{c^8}\right)$$

$$g_{0i} = -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i + \mathcal{O}\left(\frac{1}{c^7}\right)$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2\right) + \frac{4}{c^4}\hat{W}_{ij} + \mathcal{O}\left(\frac{1}{c^6}\right)$$

V, V_i, $\hat{W}_{ij}$, $\hat{R}_i$, $\hat{X}$ hierarchy of retarded potentials

example: 2 typical potentials

$$V = \Box_{\mathcal{R}}^{-1}(-4\pi G\sigma) = -\Phi_{\text{Newt.}} + \mathcal{O}\left(\frac{1}{c^2}\right)$$

$$\hat{R}_i = \Box_{\mathcal{R}}^{-1}\left[-4\pi G(V\sigma_i - V_i\sigma) - 2\partial_k V\partial_i V_k - \frac{3}{2}\partial_t V\partial_i V\right]$$

compact source   quad. source

Explicit evaluation of the potentials at a given order I

1. Expansion of the retardations of $\Box_{\mathcal{R}}^{-1} S$ in powers of $1/c$
   $\longleftrightarrow$ performable as if the source S were compact at 2.5PN
2. computation of $\int d^3\mathbf{x}' \; |\mathbf{x} - \mathbf{x}'|^{m-1} S(\mathbf{x}', t)$
   - potentials with compact source: obtained from the formulas

$$\int d^3\mathbf{x} \; F(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{y}_A) = (F)_A \qquad \text{(monopoles)}$$

$$\int d^3\mathbf{x} \; F(\mathbf{x}, t)\partial_\mu\delta(\mathbf{x} - \mathbf{y}_A) = -(\partial_\mu F)_A + \delta^0_\mu \partial_0 (F)_A \qquad \text{(dipoles)}$$

   - potentials with quadratic source: obtained by means of the kernel $g = \ln(r_1 + r_2 + r_{12})$ such that $\Delta g = 1/(r_1 r_2)$

Explicit evaluation of the potentials at a given order II

Poisson integral of $\partial_k V \partial_i V_k$

- explicit expression of the source

$$G^2 m_1 \varepsilon_{kli} \frac{n_1^k}{r_1^5} S_1^l + G^2 m_2 \varepsilon_{klm} \partial_k \left(\frac{1}{r_2}\right) \partial_{im} \left(\frac{1}{r_1}\right) S_1^l + 1 \leftrightarrow 2$$

- rewriting of the self and interaction terms using $\partial_i r_1^p = -\partial_{1i} r_1^p$ $n_1^k / r_1^5 = \partial_{1k}(4r_1^4)^{-1}$ and $\partial_k r_2^{-1} \partial_{im} r_1^{-1} = -\partial_{2k} r_2^{-1} \partial_{1im} r_1^{-1}$
- computation of the elementary Poisson integrals $\Delta^{-1} r_1^{-4} = r_1^{-2}/2$ and $\Delta^{-1}(r_1 r_2)^{-1} = g + \text{const}$

3. Derivation of $g_{\mu\nu}$ and $T^{\mu\nu}$ needed for the next order $+$ evolution equations for elimination of higher order ∂_t's
4. Iteration of the whole procedure

2.5PN equations of motion and precession equations

Papapetrou equations neglecting SS interactions [Papapetrou (1951)]

- equations of motion $\Leftrightarrow$ integral form of $\nabla_\nu T^{\mu\nu} = 0$

$$m_A c \frac{Du^\mu}{d\tau_A} = -\frac{1}{2} S_A^{\lambda\rho} u_A^\nu R^\mu_{A\,\nu\lambda\rho} + \mathcal{O}(S^2/c^4)$$

deviation from the geodesic motion due to SO interaction

- equations of precession $\leftarrow$ SSC & link $p_A^\mu / S_A^{\mu\nu}$

$$\frac{DS_A^{\mu\nu}}{c d\tau_A} = \mathcal{O}(S^2/c^4)$$

parallel transport of the spin tensor

PN equations for a self-gravitating system derived in 3 steps

- insertion of the 2PN metric in the Papapetrou equations at $\mathbf{x}$
- Hadamard regularization at $\mathbf{y}_A$
- elimination of higher order $d/dt \longrightarrow \dot{\mathbf{v}}_A = \mathbf{a}_A(r_{12}, \mathbf{n}_{12}, \mathbf{v}_B, \mathbf{S}_B)$
  idem for $\dot{\mathbf{S}}_A$

Conserved quantities

computation of the binding energy E

1. gravitational radiation switched off
2. E decomposed as the sum of $E_{2\mathrm{PN}} + E_{(5)\mathrm{SO}}/c^5$
   - $E_{2\mathrm{PN}}$ including the 1.5PN SO interactions already known
     SS contributions ignored
   - $E_{(5)\mathrm{SO}}$ searched under the form

$$E_{(5)\mathrm{SO}} = (S_1, n_{12}, v_1)(\alpha_1^{(v_1^4)} v_1^4 + \alpha_1^{v_1^2(v_1 v_2)} v_1^2 (v_1 v_2) + \ldots)/r_{12}$$
$$+ \text{all other terms hom. to } S_1 v^5/r \text{ \& invariant under trans./rot./parity} + 1 \leftrightarrow 2$$

3. dE/dt imposed to be 0
   $\hookrightarrow$ coefficients α determined uniquely

Similar computation for linear momentum $\mathbf{P}$, angular momentum $\mathbf{J} = \mathbf{L} + (\mathbf{S}_1 + \mathbf{S}_2)/c$ and $\mathbf{G}$ / $d\mathbf{G}/dt = \mathbf{P}$

Reduction to circular motion

ϟ impossibility of rigorously circular motion for spinning binaries ϟ

in CM $\mathbf{L} \propto \mathbf{n}_{12} \times (\mathbf{v}_1 - \mathbf{v}_2) \perp$ orbital plane

$\mathbf{S}_A$ precession about $\mathbf{J} \Rightarrow$ precession of $\mathbf{L} = \mathbf{J} - \mathbf{S}_1/c - \mathbf{S}_2/c$

precession of the orbital plane

definition of a circular-like motion

- $r_{12} = r = \text{cst}$
- $\langle$ component (aJ) responsible for the precession $\rangle = 0$

Reduction to circular motion in 3 steps

1. CM frame imposed by $\mathbf{G} = m_1 \mathbf{y}_1 + m_2 \mathbf{y}_2 + \ldots \equiv \mathcal{O}(1/c^6)$
   $\hookrightarrow$ investigation on relative motion sufficient
   Notation: $\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$, $\mathbf{n} = \mathbf{x}/r$, $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$
2. (nv) taken to be zero
3. v^2 replaced by its value as a function of r using the EOM

Radiative metric and GW flux

- Gravitational wave-form in the radiative zone $R \gg \lambda$
  gauge invariance of the TT part of $\delta g_{\mu\nu}$ at rad. future inf. $\mathcal{I}^+$

first term of the multipole expansion of the form

$$h_{ij}^{\mathrm{TT}}(\mathbf{X}, T) = \frac{4G}{c^4 R} P_{ijab}^{\mathrm{TT}}(\mathbf{N}) \sum_{\ell=2}^{+\infty} \frac{1}{c^{\ell-2}\ell!} \Big\{ N_{i_1} \dots N_{i_{\ell-2}} U_{abi_1\dots i_{\ell-2}} - \frac{2\ell}{c(2\ell+1)} N_c N_{i_1} \dots N_{i_{\ell-2}} \epsilon_{cd(a} V_{b)i_1\dots i_{\ell-2}} \Big\}(T - R/c)$$

with $U_{i_1\dots i_\ell}$, $V_{i_1\dots i_\ell}$ = radiative multipole moments

- Flux expression (// electromagnetism)

$$\mathcal{L} = \frac{c^3}{32\pi G} \int dS \partial_t h_{jk}^{\mathrm{TT}} \partial_t h_{jk}^{\mathrm{TT}}$$

Link with the source multipole moments

qualitative definition

set of multipole moments $I_{i_1 \ldots i_\ell}$, $J_{i_1 \ldots i_\ell}$ parameterizing the vacuum linear metric in harmonic gauge

consequences

- far-zone $h^{\mu\nu}$ obtained by solving iteratively post-linear EE
  $\Rightarrow h^{\mu\nu} = \sum_{n=1}^{+\infty} G^n h^{\mu\nu}_{(n)}$ parametrized by $I_{i_1 \ldots i_\ell}$, $J_{i_1 \ldots i_\ell}$ and some gauge transformation
- link between $U_{i_1 \ldots i_\ell}$, $V_{i_1 \ldots i_\ell}$ and the source multipole moments
- relation between $I_{i_1 \ldots i_\ell}$, $J_{i_1 \ldots i_\ell}$ and the source $|g| T^{\mu\nu} + \Lambda^{\mu\nu}$ known (asymptotic matching)

implications for computing the SO terms at 2.5PN

sufficient to compute $(I_{ij})^{\rm SO}_{\rm 2.5PN}$, $(J_{ij})^{\rm SO}_{\rm 1.5PN}$, $(I_{ijk})^{\rm SO}_{\rm 1.5PN}$, $(J_{ijk})^{\rm SO}_{\rm 0.5PN}$

Approximate expression of the source multipole moments

Symmetric and Trace-Free (STF) part of $x_i x_j$

$$I_{ij} = \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x}\, |\mathbf{x}|^B \left\{ \hat{x}_{ij}\, \Sigma + \frac{1}{14c^2} \hat{x}_{ij} |\mathbf{x}|^2\, \ddot{\Sigma} + \frac{1}{504c^4} \hat{x}_{ij}\, |\mathbf{x}|^4\, \overset{(4)}{\Sigma} - \frac{20}{21c^2}\, \hat{x}_{ijk}\, \dot{\Sigma}_k - \frac{10}{189c^4}\, \hat{x}_{ijk}\, |\mathbf{x}|^2\, \dddot{\Sigma}_k + \frac{5}{54c^4}\, \hat{x}_{ijkl} \ddot{\Sigma}_{kl} \right\} + \mathcal{O}\left(\frac{1}{c^6}\right)$$

$$J_{ij} = \underset{B=0}{\mathrm{FP}}\, \varepsilon_{ab\langle i} \int d^3\mathbf{x}\, |\mathbf{x}|^B \left\{ \hat{x}_{j\rangle a}\, \Sigma_b + \frac{1}{14c^2} \hat{x}_{j\rangle a}\, |\mathbf{x}|^2\, \ddot{\Sigma}_b - \frac{5}{28c^2}\, \hat{x}_{j\rangle ac}\, \dot{\Sigma}_{bc} \right\} + \mathcal{O}\left(\frac{1}{c^4}\right)$$

- $\Sigma_{\mu\nu} = \Sigma_{\mu\nu}[\sigma_{\alpha\beta}, V, V_i, \hat{W}_{ij}, \hat{R}_i, \hat{X}]$
- Source regularized by inserting a kernel $|\mathbf{x}|^B$, $B \in \mathbb{C}$
  Integral over source = some function of B, $I(B)$ extended by analytic continuation

Definition

$\mathrm{FP}_{B=0} I(B)$ = limit $B \to 0$ of the non polar part of $I(B)$

$\hookrightarrow$ cure of the divergences at infinity

Spin terms in the source quadrupole moments I

1. Source with compact support: computed by using the properties of $\delta(\mathbf{x}-\mathbf{y}_A)$, $\partial_\mu\delta(\mathbf{x}-\mathbf{y}_A)$
2. Source with quadratic support: obtained by means of the elementary integral

$$Y_L(\mathbf{y}_1,\mathbf{y}_2) = -\frac{1}{2\pi}\mathrm{FP}\int d^3\mathbf{x}|\mathbf{x}|^B \mathrm{STF}_{i_1\ldots i_\ell} x^{i_1}\ldots x^{i_\ell}/(r_1 r_2)$$

$$= \mathrm{STF}_{i_1\ldots i_\ell}\frac{r_{12}}{\ell+1}\sum_{p=0}^{\ell} y_1^{i_1}\ldots y_1^{i_p} y_2^{i_{p+1}}\ldots y_2^{i_\ell}$$

Spin terms in the source quadrupole moments II

Working on the source

- expression of the source simplified by integration by part
  $\longleftrightarrow$ surface terms discarded with great care
- explicit expression of source pieces of the form $\hat{x}^L \partial_{i\ldots} r_1^{-1} \partial_{j\ldots} r_2^{-1}$
- rewriting of the interaction terms using $\partial_i r_1^p = -\partial_{1i} r_1^p$
- FP regularization $\longrightarrow$ possibility of permuting $\int$ and ∂_{1i}
  $\hookrightarrow$ result expressed solely in terms of the function Y_L

Typical result

$$\underset{\mathrm{S}}{J}{}_{ij}^{(\mathrm{NC})} = \frac{Gm_2}{c^3} \varepsilon_{kl\langle i} \left\{ -\varepsilon_{kmn} S_1^m \underset{2}{\partial_k} \underset{1}{\partial_{ln}} Y_{j\rangle k} + \varepsilon_{lmn} S_1^m \underset{2}{\partial_k} \underset{1}{\partial_{kn}} Y_{j\rangle k} \right\} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^5}\right)$$

Flux and energy as a function of ω

- absence of non-linear effects $\rightarrow$

$$\mathcal{L} = \frac{G}{c^5}\left\{\frac{1}{5}\dddot{I}_{ij}\dddot{I}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}\ddddot{I}_{ijk}\ddddot{I}_{ijk} + \frac{16}{45}\dddot{J}_{ij}\dddot{J}_{ij}\right] + \frac{\ddddot{J}_{ijk}\ddddot{J}_{ijk}}{84c^4}\right\}$$

$+$ terms not contributing to the spins at 2.5PN order

after reduction to the *center of mass* for *circular orbits*
$\omega = \omega(r)$ and $\mathcal{L} = \mathcal{L}(r)$ known $\rightarrow$ $\mathcal{L} = \mathcal{L}(x)$ with $x = (Gm\omega/c^3)^{2/3}$

- $\omega = \omega(r)$ and $E = E(r)$ known $\rightarrow$ $E = E(x)$

result dependence

total mass m, mass difference δm, $\nu = \mu/m$, x, the total spin $\mathbf{S}$, $\mathbf{\Sigma} \equiv m(\mathbf{S}_2/m_2 - \mathbf{S}_1/m_1)$

Results for $\mathcal{L}(x)$, $E(x)$, and $\dot\omega$ [gr-qc/0605139, gr-qc/0605140, to appear in PRD]

$$E = -\frac{\mu c^2 x}{2}\left\{1 + x\left(-\frac{3}{4} - \frac{\nu}{12}\right) + x^2\left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right) + \frac{x^{3/2}}{G\, m^2}\left[\frac{14}{3}S_z + 2\frac{\delta m}{m}\Sigma_z\right]\right.$$
$$\left. + \frac{x^{5/2}}{G\, m^2}\left[\left(13 - \frac{49}{9}\nu\right)S_z + \left(5 - \frac{8}{3}\nu\right)\frac{\delta m}{m}\Sigma_z\right] + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}$$

$$\mathcal{L} = \frac{32}{5}\frac{c^5}{G}x^5\nu^2\left\{1 + x\left(-\frac{1247}{336} - \frac{35}{12}\nu\right) + 4\pi x^{3/2} + x^2\left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)\right.$$
$$+ \pi x^{5/2}\left(-\frac{8191}{672} - \frac{583}{24}\nu\right) + \frac{x^{3/2}}{G\, m^2}\left[-4S_z - \frac{5}{4}\frac{\delta m}{m}\Sigma_z\right]$$
$$\left. + \frac{x^{5/2}}{G\, m^2}\left[\left(\frac{95}{28} + \frac{239}{63}\nu\right)S_z + \left(\frac{31}{16} - \frac{109}{28}\nu\right)\frac{\delta m}{m}\Sigma_z\right] + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}$$

$$\frac{\dot\omega}{\omega^2} = \frac{96}{5}\nu x^{5/2}\left\{1 + x\left(-\frac{743}{336} - \frac{11}{4}\nu\right) + 4\pi x^{3/2} + x^2\left(\frac{34103}{18144} + \frac{13661}{2016}\nu + \frac{59}{18}\nu^2\right)\right.$$
$$+ \pi x^{5/2}\left(-\frac{4159}{672} - \frac{189}{8}\nu\right) + \frac{x^{3/2}}{G\, m^2}\left[-\frac{47}{3}S_z - \frac{25}{4}\frac{\delta m}{m}\Sigma_z\right]$$
$$\left. + \frac{x^{5/2}}{G\, m^2}\left[\left(-\frac{40127}{1008} + \frac{1465}{28}\nu\right)S_z + \left(-\frac{583}{42} + \frac{3049}{168}\nu\right)\frac{\delta m}{m}\Sigma_z\right] + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}$$

Relevance of the SO effects at 2.5PN order I

Table: Post-Newtonian contributions to the number of GW cycles accumulated from $\omega_{\mathrm{min}} = \pi \times 10\,\mathrm{Hz}$ to $\omega_{\mathrm{max}} = \omega_{\mathrm{ISCO}} = 1/(6^{3/2}\,m)$ for binaries detectable by LIGO and Virgo. For comparison, we add the contributions of spin-spin terms at 2PN order and non-spin terms at 3PN and 3.5PN orders

Notation: $S_1 = Gm_1^2\chi_1$, $\kappa_1 = (S_1 L)/L$, $S_1 S_2 \xi = (S_1 S_2)$

	$(10+1.4)M_\odot$	$(10+10)M_\odot$
Newtonian	3577	601
1PN	$+213$	$+59.3$
1.5PN	$-181 + 114\,\kappa_1\,\chi_1 + 11.8\,\kappa_2\,\chi_2$	$-51.4 + 16.0\,\kappa_1\,\chi_1 + 16.0\,\kappa_2\,\chi_2$
2PN	$+9.8 - 4.4\,\kappa_1\,\kappa_2\,\chi_1\,\chi_2 + 1.5\,\xi\,\chi_1\,\chi_2$	$+4.1 - 3.3\,\kappa_1\,\kappa_2\,\chi_1\,\chi_2 + 1.1\,\xi\,\chi_1\,\chi_2$
2.5PN	$-20 + 32.7\,\kappa_1\,\chi_1 + 3.2\,\kappa_2\,\chi_2$	$-7.1 + 6.0\,\kappa_1\,\chi_1 + 6.0\,\kappa_2\,\chi_2$
3PN	$+2.3$	$+2.2$
3.5PN	-1.8	-0.8

Relevance of the SO effects at 2.5PN order II

Table: post-Newtonian contributions to the number of GW cycles accumulated until $\omega_{\rm max} = \omega_{\rm ISCO}$ over one year of integration, for binaries detectable by LISA

Notation: $S_1 = Gm_1^2\chi_1$, $\kappa_1 = (S_1 L)/L$, $S_1 S_2 \xi = (S_1 S_2)$

	$(10^6 + 10^6)M_\odot$	$(10^6 + 10^5)M_\odot$
Newt.	2267	4985
1PN	$+134$	$+281$
1.5PN	$-92.4 + 28.8\,\kappa_1\,\chi_1 + 28.8\,\kappa_2\,\chi_2$	$-243 + 161\,\kappa_1\,\chi_1 + 11.5\,\kappa_2\,\chi_2$
2PN	$6.0 - 4.8\,\kappa_1\,\kappa_2\,\chi_1\,\chi_2 + 1.7\,\xi\,\chi_1\,\chi_2$	$12.5 - 4.4\,\kappa_1\,\kappa_2\,\chi_1\,\chi_2 + 1.5\,\xi\,\chi_1\,\chi_2$
2.5PN	$-9.0 + 7.6\,\kappa_1\,\chi_1 + 7.6\,\kappa_2\,\chi_2$	$-26.5 + 44.8\,\kappa_1\,\chi_1 + 3.0\,\kappa_2\,\chi_2$
3PN	$+2.3$	$+2.3$
3.5PN	-0.9	-2.3

Consequences

data analysis with SO corrections at 2.5PN needed

for some mass configurations: SS effects at 2PN $\lesssim$ SO at 2.5PN

Spin variables with constant magnitude

Existence of a SSC

arbitrariness in the spin definition

$\Rightarrow$ possibility to redefine the spin : chosen such that $S_1^2 = \text{const}$

Consequences

- $\mathbf{S}_1^c = \mathbf{S}_1 + \frac{1}{c^2}\left[-\frac{1}{2}(v_1 S_1)\mathbf{v}_1 + \frac{Gm_2}{r_{12}}\mathbf{S}_1\right] + \text{2PN corrections} + \mathcal{O}\left(\frac{1}{c^5}\right)$
- new orbital momentum $\mathbf{L}^c = \mathbf{J} - \mathbf{S}_1^c/c - \mathbf{S}_2^c/c$
- precession equations of the form

$$\frac{d\mathbf{S}_1^c}{dt} = \mathbf{\Omega}_1 \times \mathbf{S}_1^c \qquad \frac{d\mathbf{S}_2^c}{dt} = \mathbf{\Omega}_2 \times \mathbf{S}_2^c$$

Conclusion

- GW phase obtained by
  1. Integration of the system giving $d\omega/dt$, dS_1^c/dt, dS_2^c/dt
  2. addition of the well-known precessional correction
- Use of spins with constant magnitude recommended
  $\leftarrow$ equivalence with the template space built with waveforms involving the variables $\mathbf{S}_1$ and $\mathbf{S}_2$
- Crude estimation suggests the importance of the SO terms in the 2.5PN phase
- Seems that spins can be determined rather accurately with the 2.5PN corrections
  ν loses accuracy [Arun, Buonanno, F., Ochsner, in progress]

Open issues: non linear spin effects, 2.5 amplitudes with spins, ...