WELL-POSEDNESS AND NUMERICAL STABILITY OF THE GLOBAL HARMONIC INITIAL VALUE PROBLEM

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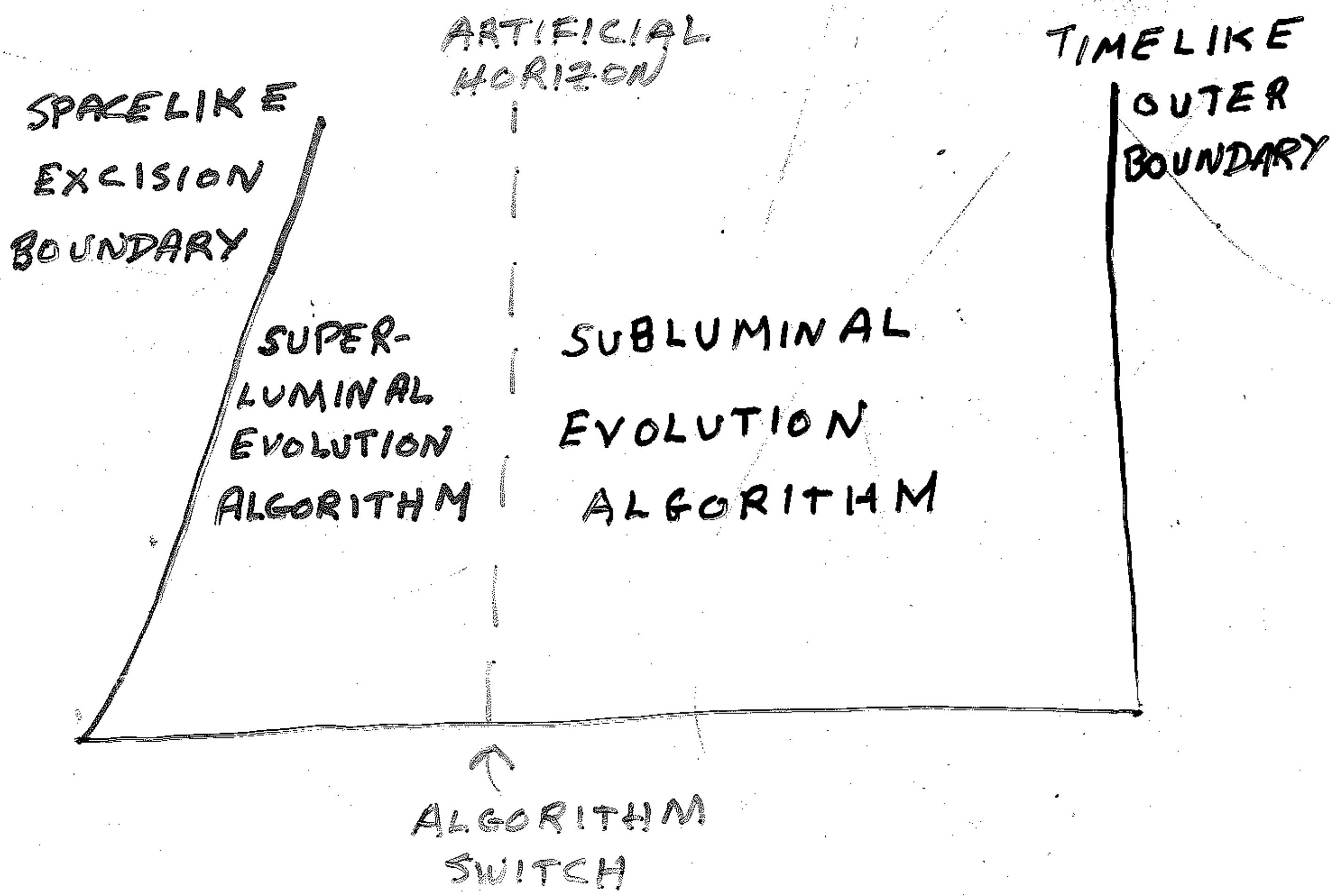
I discuss theorems which establish the well-posedness and computational stability of harmonic evolution and boundary algorithms for a black hole spacetime. Of particular importance, results regarding constraint preserving Sommerfeld boundary conditions can be used to enhance the computation of gravitational waveforms by matching techniques.

THE INITIAL-BOUNDARY VALUE PROBLEM FOR BLACK HOLE EXCISION



Finite characteristic speeds allow global problem to be reduced to half-space and Cauchy problems.

REFERENCE MATERIAL

M. C. Babiuc, H-O. Kreiss and J. Winicour (in preparation) "Constraint-preserving Sommerfeld conditions for the harmonic Einstein equations"

H-O. Kreiss and J. Winicour (2006) "Problems which are well-posed in a generalized sense with applications to the Einstein equations"

M. Motamed, M.C. Babiuc, B.Szilàgyi, H-O. Kreiss and J. Winicour (2006) "Finite difference schemes for second order systems describing black holes"

M. C. Babiuc, B. Szilàgyi and J. Winicour (2006) "Testing numerical evolution with the shifted gauge wave"

M. C. Babiuc, B. Szilàgyi and J. Winicour (2006) "Harmonic Initial-Boundary evolution in general relativity"

B. Szilàgyi, B. and J. Winicour (2003) "Well-posed initial-boundary evolution in general relativity"

HARMONIC EVOLUTION

Evolution variables: $\gamma^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$

Evolution equations:

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\gamma^{\mu\nu} = S^{\mu\nu}$$

Constraints:

$$C^\mu := -\frac{1}{\sqrt{-g}}\partial_\mu\gamma^{\mu\nu} - \hat{\Gamma}^\nu(g,x) = 0$$

Gauge source functions $\hat{\Gamma}^\nu$ do not enter principle part.

Advantages for numerical and analytic studies:

- **Small number of variables**
- **Small number of constraints (4 harmonic conditions)**
- **Quasilinear wave equations**
- **Well-posed Cauchy problem (Choquet-Bruhat)**
- **IN ADDITION: Well-posed IBVP and stable evolution and boundary algorithms for black hole simulation**

Constraint preservation:

$$g^{\alpha\beta}\partial_\alpha\partial_\beta C^\mu + A^{\mu\alpha}_\beta\partial_\alpha C^\beta + B^\mu_\beta C^\beta = 0$$

Uniqueness implies $C^\mu = 0$ provided
$C^\mu|_{t=0} = \partial_t C^\mu|_{t=0} = 0$ **and** $C^\mu|_{\text{Boundary}} = 0$

The Evolution Algorithms (2D Scalar)

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\Phi = S$$

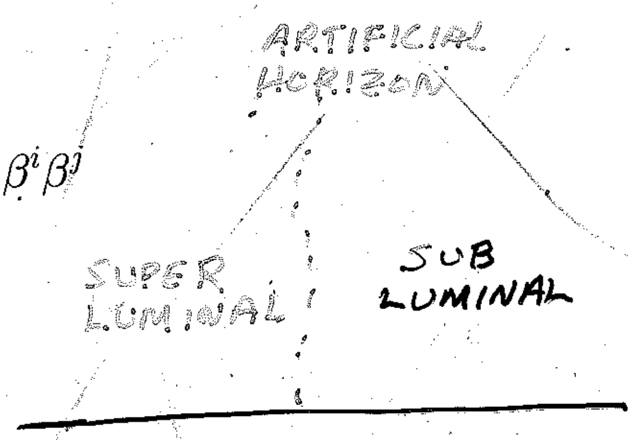
The subluminal algorithm:

$$W := \left(g^{tt}\partial_t^2 + 2g^{ti}\partial_t D_{0i} + g^{xx}D_{+x}D_{-x} + g^{yy}D_{+y}D_{-y} + 2g^{xy}D_{0x}D_{0y}\right)\Phi$$

Method of lines (Runge-Kutta)
Subluminal if

$$g^{ij} = h^{ij} - \frac{1}{\alpha^2}\beta^i\beta^j$$

has Euclidean signature.



Artificial horizon: $\det(g^{ij}) = 0$
W**-algorithm is stable if** $g^{ii} > 0$.

The superluminal algorithm:

$$V_\alpha := W - \frac{h^2}{4}\left(\alpha_x(D_{+x}D_{-x})^2 + \alpha_y(D_{+y}D_{-y})^2\right)\Phi$$

Here α_i **are "switches" which modify the algorithm inside the artificial horizon, e.g.**

$$\alpha_x = \frac{1}{2}\left(|g^{xx}| - g^{xx}\right)$$

V_α **algorithm is everywhere stable, but less accurate outside the artificial horizon.**

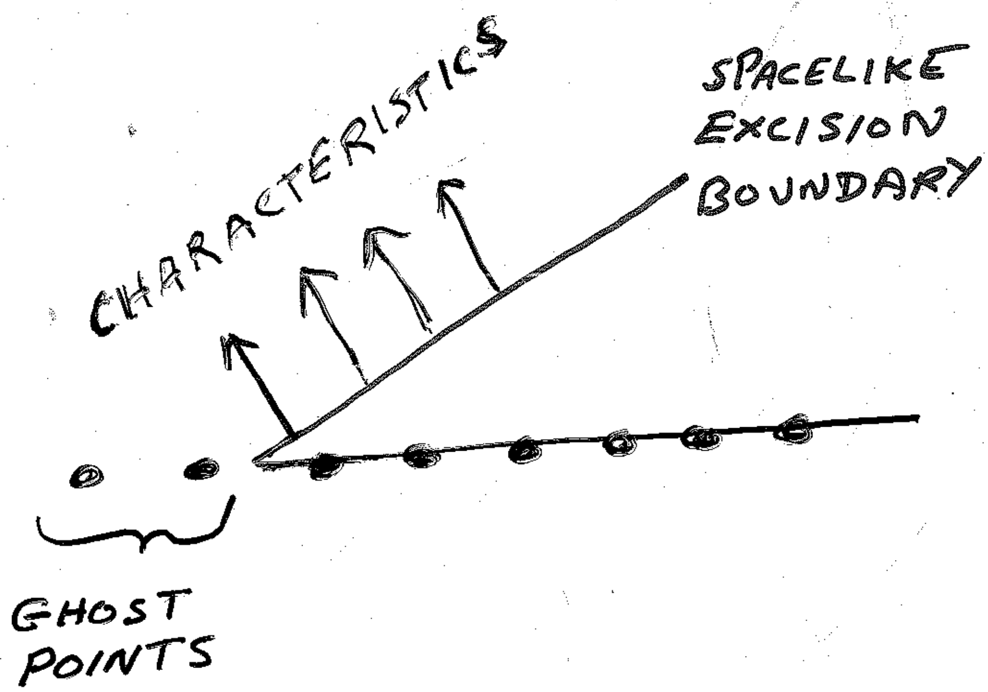
THE SPACELIKE EXCISION BOUNDARY

All characteristics leave the domain – pure outflow.

CONSEQUENCES:

- No boundary condition or boundary data allowed.
- Numerical noise leaves the grid.
- BLACK HOLES ARE BENEVOLENT.

Stable extrapolation algorithm to fill "ghost points" in superluminal V_α-algorithm.



TIMELIKE OUTER BOUNDARY - ENERGY METHOD

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\Phi = S.$$

Maximally dissipative boundary conditions - energy flux:

$$\mathcal{F} = -(\partial_t\Phi)n^\alpha\partial_\alpha\Phi$$

where n^α is normal to boundary.

The IBVP for the general scalar wave equation is well-posed for any of the following boundary conditions:

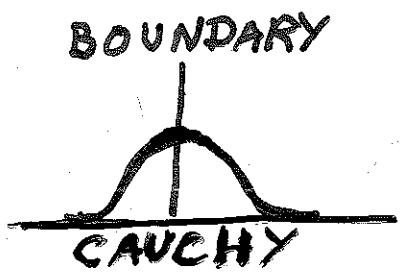
- **Dirichlet:** $\partial_t\Phi = q$, $\quad q = 0$ implies $\mathcal{F} = 0$
- **Neumann:** $n^\alpha\partial_\alpha\Phi = q$, $\quad q = 0$ implies $\mathcal{F} = 0$
- **Sommerfeld:** $\partial_t + n^\alpha\partial_\alpha = q$, $\quad q = 0$ implies $\mathcal{F} > 0$ **(outgoing null direction)**

Discrete energy method (SBP): The W-algorithm is numerically stable for above boundary conditions. Only Sommerfeld lets the noise escape.

HARMONIC CONSTRAINTS

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\gamma^{\mu\nu} = S^{\mu\nu}, \quad \mathcal{C}^\nu := \partial_\mu\gamma^{\mu\nu} = 0$$

REFLECTION SYMMETRY $\rightarrow$ well-posed CONSTRAINT PRESERVING Dirichlet/Neumann boundary conditions.



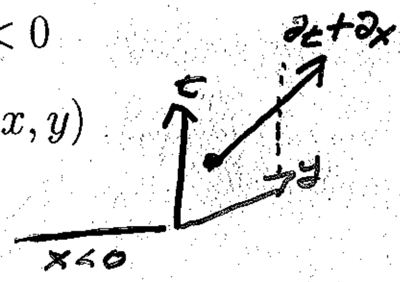
Friedrich-Nagy system first well-posed constraint-preserving boundary conditions of Sommerfeld type.

WELL-POSEDNESS IN A GENERALIZED SENSE

The energy method cannot be used to study the well-posedness of the system

$$(\partial_t^2 - \partial_x^2 - \partial_y^2)v = S, \quad -\infty \leq x < 0$$

with initial data $v|_{t=0} = f_1(x,y), \quad \partial_t v|_{t=0} = f_2(x,y)$
and Sommerfeld-like boundary condition
$\partial_t v + \partial_x v + M\partial_y v = q(t,y)$ **at** $x = 0$.



Instead pseudo-differential theory can be used to show when the problem is strongly well-posed in the generalized sense:

$$\|v\|^2 \leq K\{.... + \|\partial^2 f\|^2\}.$$

Establish estimates by constructing symmetrizer for Laplace transform in t **and Fourier transform in** y.

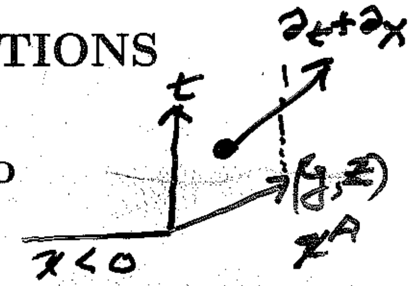
USEFUL FEATURES:

- **The principle of frozen coefficients holds.**
- **Well-posedness is insensitive to lower order terms.**
- **Applicable to second differential order systems.**
- **Derivatives can be estimated so that well-posedness extends locally in time to quasi-linear problems.**

By adapting the harmonic coordinates to the boundary and introducing an orthonormal tetrad, the method applies to the quasilinear harmonic wave equations

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\gamma^{\mu\nu} = S^{\mu\nu}$$

CONSTRAINT PRESERVING SOMMERFELD BOUNDARY CONDITIONS



With frozen coefficients, the IBVP reduces to

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta\gamma^{\mu\nu} = S^{\mu\nu}, \quad \mathcal{C}^\nu := \partial_\mu\gamma^{\mu\nu} = 0$$

in the evolution domain $x < 0$, $\quad x^\mu = (t, x, y, z) = (t, x, x^A)$.

Constraint preserving Sommerfeld boundary conditions at $x = 0$. **Simplest choice:**

First impose the 6 Sommerfeld boundary conditions

$$(\partial_t + \partial_x)\,\gamma^{AB} = q^{AB}$$

$$(\partial_t + \partial_x)(\gamma^{tA} - \gamma^{xA}) = q^A$$

$$(\partial_t + \partial_x)(\gamma^{tt} - 2\gamma^{tx} + \gamma^{xx}) = q$$

where q^{AB}, q^A **and** q **are free Sommerfeld data.**

Then use the constraints to supply 4 additional Sommerfeld boundary conditions in the hierarchical order

$$\mathcal{C}^A = \frac{1}{2}(\partial_t + \partial_x)(\gamma^{tA} + \gamma^{xA}) + \partial_t(\gamma^{tA} - \gamma^{xA}) + \partial_B\gamma^{AB} - \frac{1}{2}q^A = 0$$

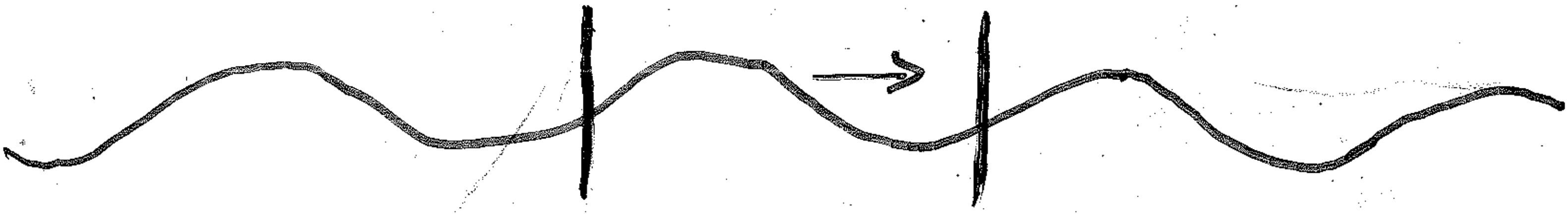
$$\mathcal{C}^t - \mathcal{C}^x = \frac{1}{2}(\partial_t + \partial_x)(\gamma^{tt} - \gamma^{xx}) + \partial_t(\gamma^{tt} - 2\gamma^{tx} + \gamma^{xx}) + \partial_B(\gamma^{tB} - \gamma^{xB}) - \frac{1}{2}q = 0$$

$$\mathcal{C}^t = \frac{1}{2}(\partial_t + \partial_x)(\gamma^{tt} + \gamma^{xx}) + \partial_t(\gamma^{tt} - \gamma^{tx}) + \partial_B\gamma^{tB} - \frac{1}{2}q = 0$$

In expressing the constraints in this form, we have used the prior boundary conditions in the hierarchy.

USING A NULL PROJECTION OPERATOR, THESE BOUNDARY CONDITIONS CAN BE APPLIED TO THE HARMONIC EINSTEIN EQUATIONS.

THE SHIFTED GAUGE WAVE TEST

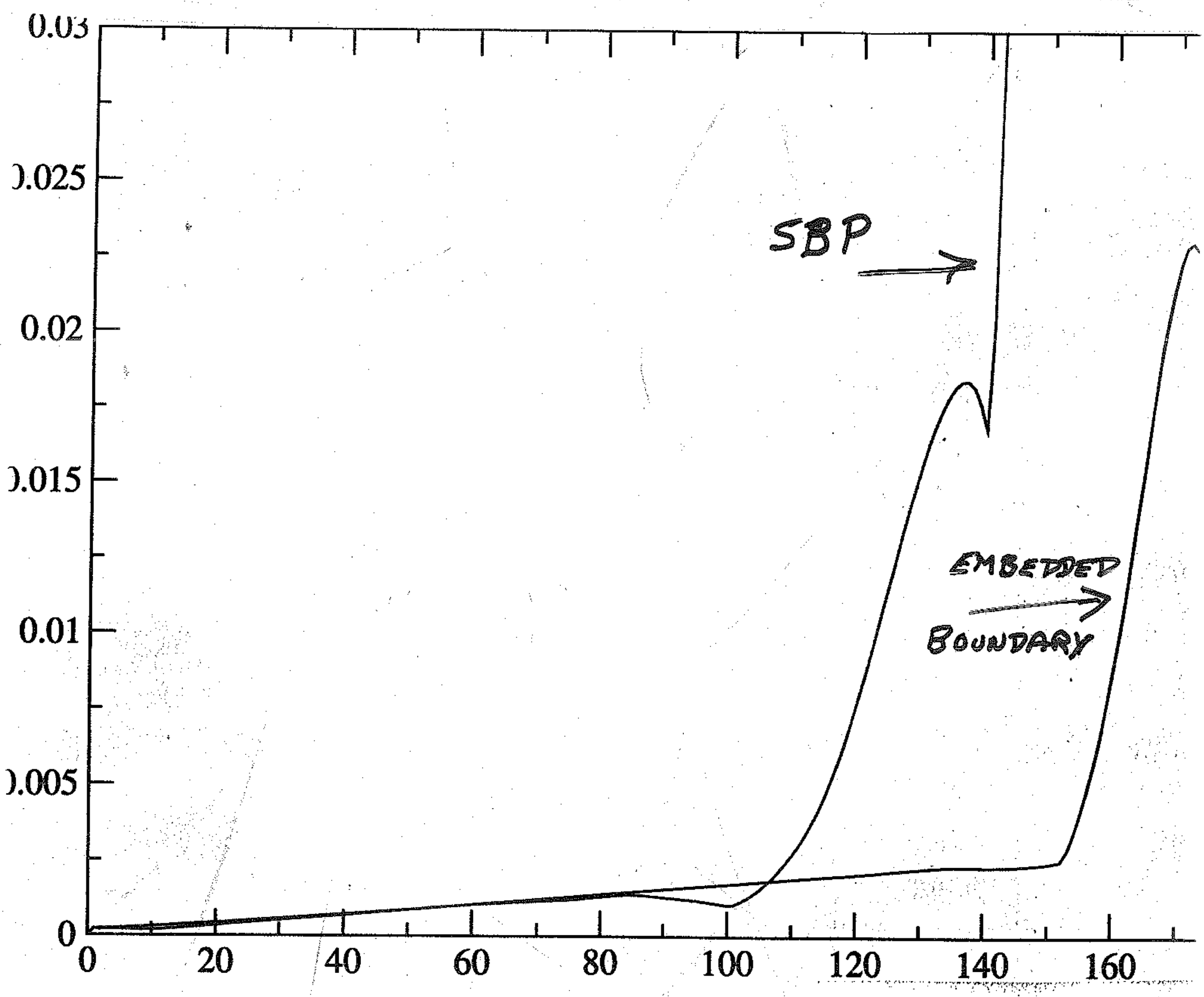


$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + H k_\alpha k_\beta dx^\alpha dx^\beta,$$

$$H = .5\sin\left(2\pi(x-t)\right), \quad k_\alpha = \partial_\alpha(x-t)$$

The analytic problem has long wavelength constraint violating instability.

CONSTRAINED SOMMERFELD - ERROR GROWTH



WHAT CAN GO WRONG WITH A MATHEMATICALLY WELL-POSED AND NUMERICALLY STABLE CONSTRAINT PRESERVING IBVP?

NUMERICALLY INACCURATE:

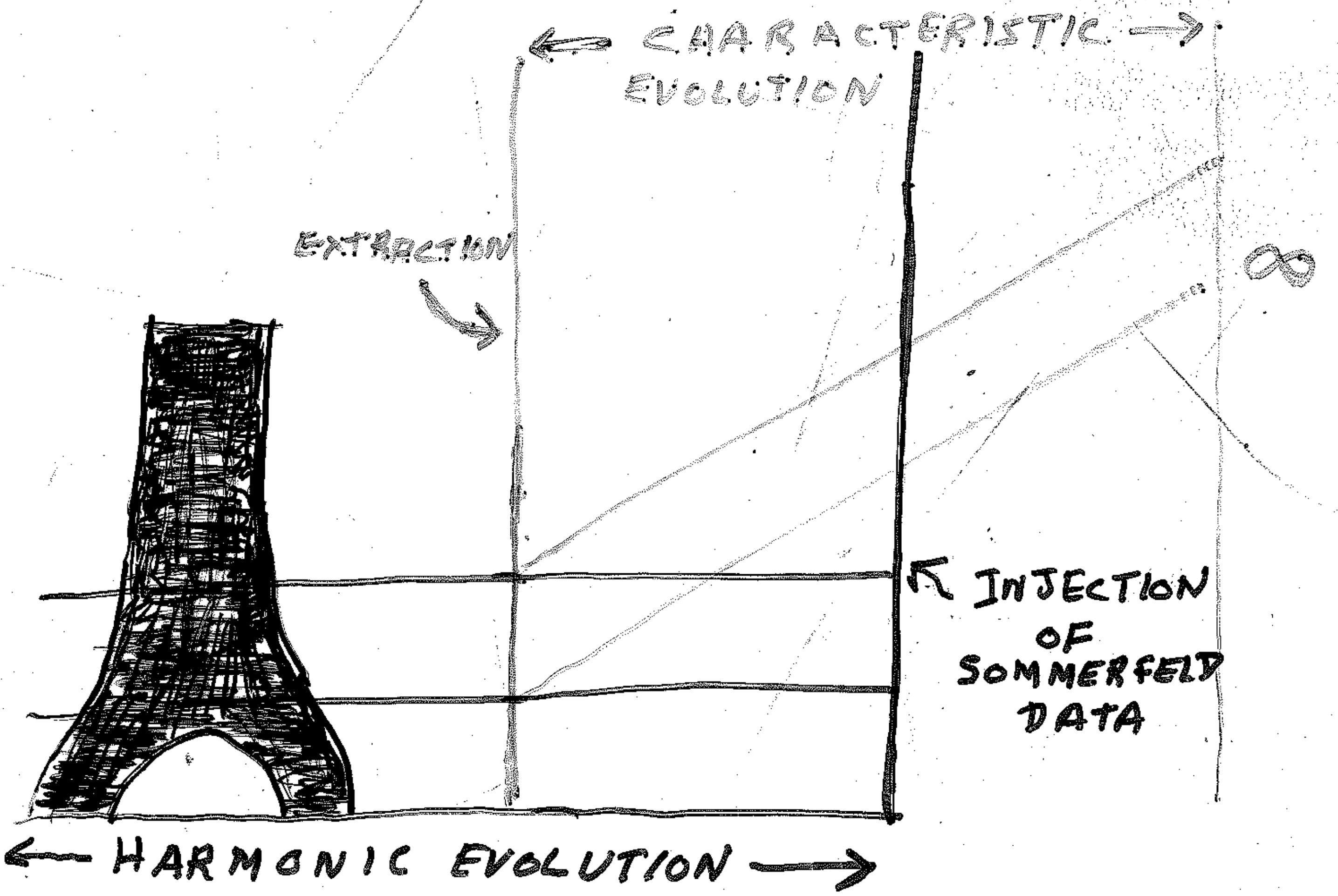
- Higher order accuracy
- Mesh refinement
- Spherical boundaries: Embedded boundary techniques (interpolation) or Multi-block methods (Summation by parts)
- Excision boundary moving through grid, common excision boundary for BBH merger

PHYSICALLY - GEOMETRICALLY:

- Singularities or instabilities in the analytic problem
- Boundary data wrong
- Can't extract waveform

GETTING THE RIGHT WAVEFORM

Extend the solution to infinity using Cauchy-characteristic matching



- Extract characteristic data for gravitational field and harmonic coordinates at inner worldtube.
- Propagate field and harmonic coordinates to infinity using characteristic code.
- Calculate Jacobian between harmonic and characteristic coordinates at outer Cauchy boundary.
- Inject constraint preserving Sommerfeld data at Cauchy boundary.
- Cauchy evolve to next time level.