Outline

1. MOTivation

2. The Conformal Method

3. Bifurcation Theory and unscaled sources

4. NON-uniqueness in the extended Conformal Thin Sandwich formulation

MOTIVATION

Pfeiffer + York '05:

(eqn Conformal Thin Sandwich) + ($\mathring{K}$)

= 5 coupled non-lin elliptic PDES for $(\phi, \tilde{N}, \beta^i)$

Initial Data

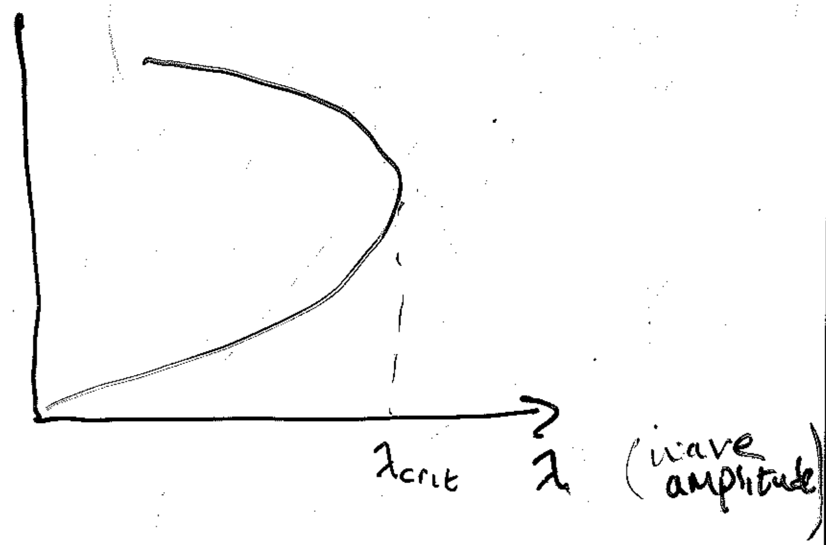
$$\tilde{\gamma}_{ij} = \delta_{ij} + \lambda h_{ij}$$
$$u_{ij} = \lambda \mathring{h}_{ij} \qquad \text{(tracefree)}$$

h_{ij} = Teukolsky wave
λ = wave amplitude

Results

$$\boxed{\vec{Y} = \vec{Y}_c \pm \sqrt{\epsilon}\,\vec{W}}$$

(Max ϕ, Min N, Max β)



?

The Conformal Method

$$G_{\mu\nu} = T_{\mu\nu}$$

$$\underbrace{\phantom{G_{\mu\nu} = T_{\mu\nu}}}$$

6 evolution eqns

4 constraint eqns

Constraints

$$R(g) - K \circ K + K^2 = 16\pi \rho$$

$$\nabla_i \left[K^{ij} - K g^{ij} \right] = 8\pi J^j$$

choose γ_{ij} s.t. $g^{phys}_{ij} = \varphi^4 \gamma_{ij}$

$$\Rightarrow R(g)\phi^5 = R(\gamma)\phi - 8\nabla^2\varphi = \phi^5(K.K - K^2 + \rho)$$

$$\left(K_{ij} = A_{ij} + \frac{g_{ij}}{3} K \quad ; \text{ take } \begin{array}{l} K = 0 \\ J^j = 0 \end{array} \right)$$

$$\Rightarrow \boxed{\begin{array}{l} \nabla^2 \phi - \frac{R}{8}\phi + \frac{\phi^5}{8}\left(A \circ A + \rho_{16\pi}\right) = 0 \\ \nabla_i A^{ij} = 0 \end{array}}$$

Choose $A_{ij} = \phi^{-10} \hat{A}_{ij}$, $\rho = \varphi^{-6} \hat{\rho}$, $K = 0$

Linearise LY : (about a solution)

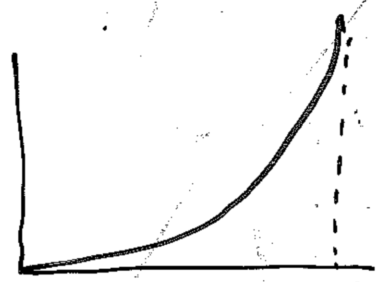
$$(\nabla^2 - f^2)\delta\phi = 0$$

↳ isomorphism (between appropriate spaces)

Imp Fn Thm ⟹ there exists unique solutions nearby

qualitatively

~ Pfeiffer + York '05



Suppose we don't scale the extrinsic curvature and/or ρ

$$\Longrightarrow (\nabla^2 + f)\delta\phi = 0$$

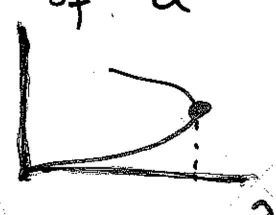
indefinite

↳ no maximum principle

... Linearisation not isomorphic

[note: Many constrained evolutions don't scale the momentum]

Proposal results of P-Y are due to bifurcation (Linearisation of a solution has a kernel) φ



Basic Elliptic PDE's

Linear

$$B u = h(x) \quad , \quad B \text{ linear + elliptic}$$

$\times v$ $\quad \int v h = \int v B u = \int (B^* v) u$

$\therefore$ $Bu = h$ is solvable

$\iff \int h v = 0$ for all $v \in \operatorname{Ker} B^*$

non-linear

- Imp. Fn. thm:

$$B u = h(u) \quad , \quad B \text{ linear in our application}$$

if B is isomorphism
then there exists unique "small" solutions.

Bifurcation Theory

<u>non-linear eqn</u> : $F : X \longrightarrow Y$

$$F = \boxed{\nabla^2\phi + \lambda\rho\phi^5 = 0}$$

<u>Assume given</u> : @ $\lambda = \lambda_c$ there's a kernel solution V_0 and critical solution ϕ_c

$$\begin{cases} \nabla^2\phi_c + \lambda_c\rho\phi_c^5 = 0 \\ \nabla^2 V_0 + 5\lambda_c\rho\phi_c^4 V_0 = 0 \end{cases}$$

<u>Expand in nbd of ϕ_c</u> :

$$\lambda = \lambda_c - \epsilon$$
$$\phi = \phi_c + V$$

$$\therefore \nabla^2\phi + \lambda\rho\phi^5 \longmapsto \nabla^2(\phi_c + V) + (\lambda_c - \epsilon)\rho(\phi_c + V)^5 = 0$$

$$\Rightarrow \nabla^2 V + \lambda_c\rho\phi_c^5 V = -\epsilon\phi_c^5\rho - 5\epsilon\phi_c^4\rho V + 10\phi_c^3\rho\lambda_c V^2 + \text{higher order terms}$$

$$B\,V = R(V, \epsilon)$$

where B has 1-D kernel,

$$B V_0 = 0$$

$$BV = R(V, \epsilon) \quad , B : X \to Y$$

Problem : B has 1-D Kernel V_0

$\therefore$ can't use IFT

Solution : remove the kernel

Split Domain $\quad V (\in X) = \gamma V_0 + u$

Split Range $\quad R (\in Y) = dZ + \omega$

$u \xrightarrow{\hat{B}} \omega$

(Define z : $\int z . V_0 = 1$, $\int u . z = 0$)

$\int \omega . V_0 = 0$

$BV = R(V, \epsilon)$ $\xrightarrow{\text{Equivalent to}}$ $\boxed{\begin{array}{l} ① \ \hat{B} u = R(\gamma V_0 + u, \epsilon) \\ ② \ \int V_0 R(\gamma V_0 + u, \epsilon) = 0 \end{array}}$

① $\hat{B}$ is bijective

↳ linearisation of ① : $\hat{B} u = 0$

IFT $\Rightarrow \exists$ solution $u = u(\gamma, \epsilon)$

② Sub $u(\gamma, \epsilon)$ into ②

$$\int V_0 R(\gamma V_0 + u(\gamma, \epsilon), \epsilon) = 0$$

(i.e. $d = 0$)

$$\Longrightarrow \gamma = \gamma(\epsilon)$$

(② is the Lyapunov-Schmidt eqn)

What have we found?

original equation:

$$BV = R(V,\epsilon) \qquad \text{s.t. } BV_0 = 0$$

we have a solution

$$\boxed{V = \underbrace{\gamma(\epsilon)}_{\text{determined by}} V_0 + \underbrace{u(\gamma,\epsilon)}_{\text{from IFT}}}$$

$$\int V_0 R(\gamma V_0 + u, \epsilon) = 0$$

Task find $\gamma = \gamma(\epsilon)$

recall $R(V,\epsilon) = -\epsilon \phi_c^5 \rho - 5\epsilon V \phi_c^4 \rho$

$$10\phi_c^3 \rho \lambda_c V^2 + \dots$$

expand V in powers of γ, ϵ

$$V = \underset{\substack{\shortparallel \\ V_0}}{V_{10}}\gamma + \epsilon V_{01} + \gamma\epsilon V_{11} + \dots$$

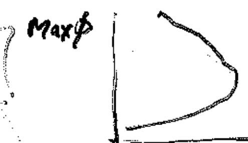
sub into $\int R V_0 = 0$:

$$\underline{\gamma^2} \underbrace{\int [\dots] V_0}_{\neq 0} + \underline{\gamma\epsilon} \int [\dots] V_0 + \underline{\epsilon} \underbrace{\int [\;] V_0}_{\neq 0} \simeq 0$$

$$\Rightarrow \boxed{\gamma = \pm (\text{const}) \sqrt{\epsilon}}$$

recall $\phi = \phi_c + V = \phi_c + \gamma V_0 + u$

$$\Rightarrow \boxed{\phi = \phi_c \pm \sqrt{\epsilon}\, V_0 + \dots}$$



Review of Pfeiffer - York '05

Comparison of 2 formulations of Constraints

① CTS : 4 eqn System:

Improved "physical" initial data:

$$(\gamma_{ij}, u_{ij}, K, N)$$ (u_{ij} = Tracefree $\dot{\gamma}_{ij}$ /

unknowns: (ψ, N, β^i)

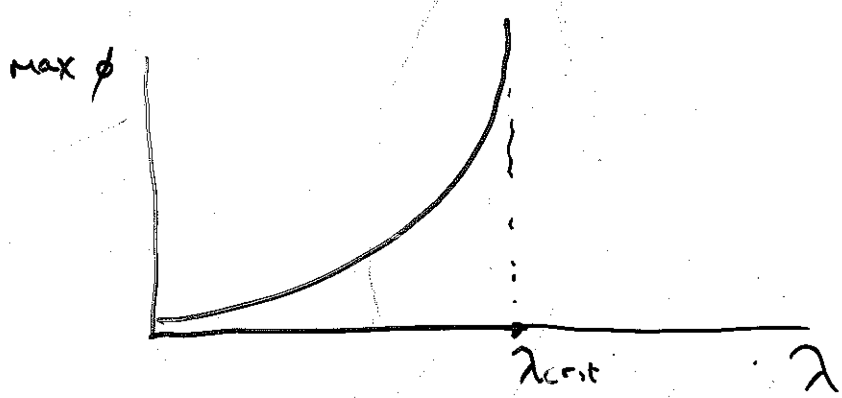
. system decouples for $K = \text{const}$.

choose $\gamma_{ij} = \delta_{ij} + \lambda h_{ij}$

$u_{ij} = \lambda h_{ij}$

λ = amplitude of Teukolsky wave

Results



- no solutions for $\lambda > \lambda_{crit}$ (as expected - Cantor + Brill '81) and unique solutions for $\lambda < \lambda_{crit}$
- $E_{ADM} \to \infty$ as $\lambda \to \lambda_{crit}$

ctd :

2 Extended conformal Thin Sandwich : (XCTS)

$$(CTS) + (\dot{K}) = XCTS$$

$$4 + 1 = 5$$

Initial data $(\gamma_{ij}, u_{ij}, K, \dot{K})$

unknowns (ϕ, β^i, N)

$$\boxed{\begin{aligned} &\nabla^2\phi - \frac{R\phi}{8} = -\frac{|\mathbb{L}\beta - u|^2}{32}\frac{\phi^7}{\chi} \\ &\nabla^2\chi - \frac{R\chi}{8} = \frac{7}{32}|\mathbb{L}\beta - u|^2\frac{\phi^6}{\chi} \\ &\nabla_i\left[\frac{\phi^7}{\chi}(\mathbb{L}\beta^{ij} - u^{ij})\right] = 0 \end{aligned}}$$

$(\chi := N\phi^7$ chosen $K = \dot{K} = 0)$

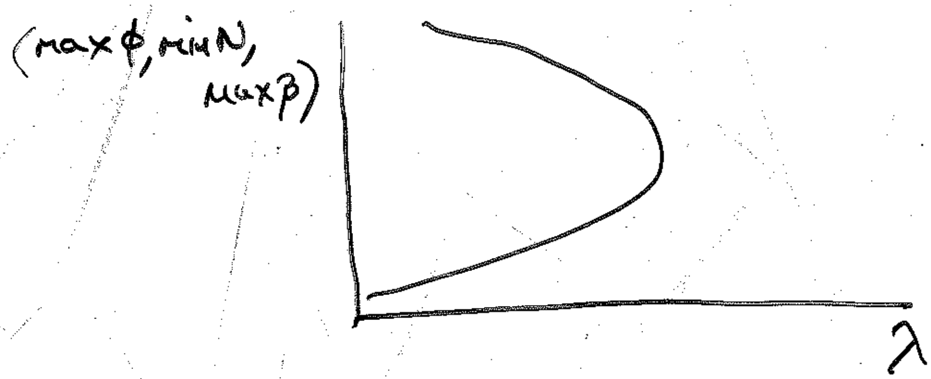
- no $K =$ const. decoupling.

Benefit : natural Quasi-Equilibrium

$$u^{ij} = K = \dot{K} = 0$$

But for the Teukolsky wave $(u_{ij} \neq 0)$

$$(\phi, N, \beta) := \vec{Y} = \vec{Y}_c \pm \sqrt{\varepsilon}\,\vec{W}, \quad \varepsilon = \delta\lambda$$



Can we Apply Lyapunov-Schmidt methods?

- A perturbation problem $\therefore$ Assume a solution Y_c exists whose linearisation has a kernel $\vec{V_0}$
- need $\text{Ker}\,\vec{B} = \text{Ker}\,\vec{B}^*$
  - this is true by examining the linearisation
- Imp Fn Theorem also valid for systems.

Proceed as before:

Split domain $\vec{V} = \gamma\vec{V_0} + u$

" Range $\vec{R} = d\vec{z} + \vec{\omega}$

$\left(\int u.z = 0 \;,\; \int \vec{\omega}.\vec{V_0} = 0 \;, \ldots \quad \text{as before} \right)$

Projection:

$$\hat{B}\vec{u} = \vec{R}(\gamma V_0 + \vec{u}, \epsilon) \qquad ①$$

$$\int \vec{V_0}.\vec{R}(\gamma\vec{V_0} + \vec{u}, \epsilon) = 0 \qquad ②$$

Imp Fn Thm: There exists unique small $\vec{u} = \vec{u}(\gamma, \epsilon)$

Sub into ② $\Rightarrow \gamma = \gamma(\epsilon)$

$\therefore \; B\vec{V} = R(\vec{V})$ has solution

$$\vec{V} = \gamma(\epsilon)\vec{V_0} + \vec{u}$$

Taylor expansion of $\vec{V}$ gives $\gamma = \pm\sqrt{\epsilon}$

(assuming very general integrals don't vanish)

Then solution of non-linear problem is

$$\boxed{\vec{Y} = \vec{Y_c} \pm \sqrt{\epsilon}\,\vec{V_0} + \ldots}$$

$\rightarrow$ similar to Pfeiffer + York '05

What have we shown?

- results of P-Y suggest that the linearised system has a kernel at the critical wave amplitude where simple bifurcation effects occur
- Basic non-linear methods can explain complex PDE problems
- should expect similar behaviour in any non-linear elliptic system (if linearisation about exact solution has a kernel)

Implications? (with Oliver Rinne)

- a possibility reason for the success of free rather than constrained evolutions in some formulations where A^{ij} is not rescaled.
  - $\dot\phi = \ldots$ just selects one solution
  - constrained evolution: potential problems on each slice! (If the momentum isn't scaled ... behaves like $S\phi^5$)
- P-Y results were GLOBAL
  - $\mapsto$ see Also Baumgarte, Ó Murchadha, Pfeiffer ('06) (analysis of constant density star)
- P-Y used regular initial data
  - $\rightarrow$ much more complicated non-uniqueness using XCTS with isolated horizon boundary conditions (Ansorg, Jaramillo, Lemonsan '06)