Moving puncture method: recent results and outlook

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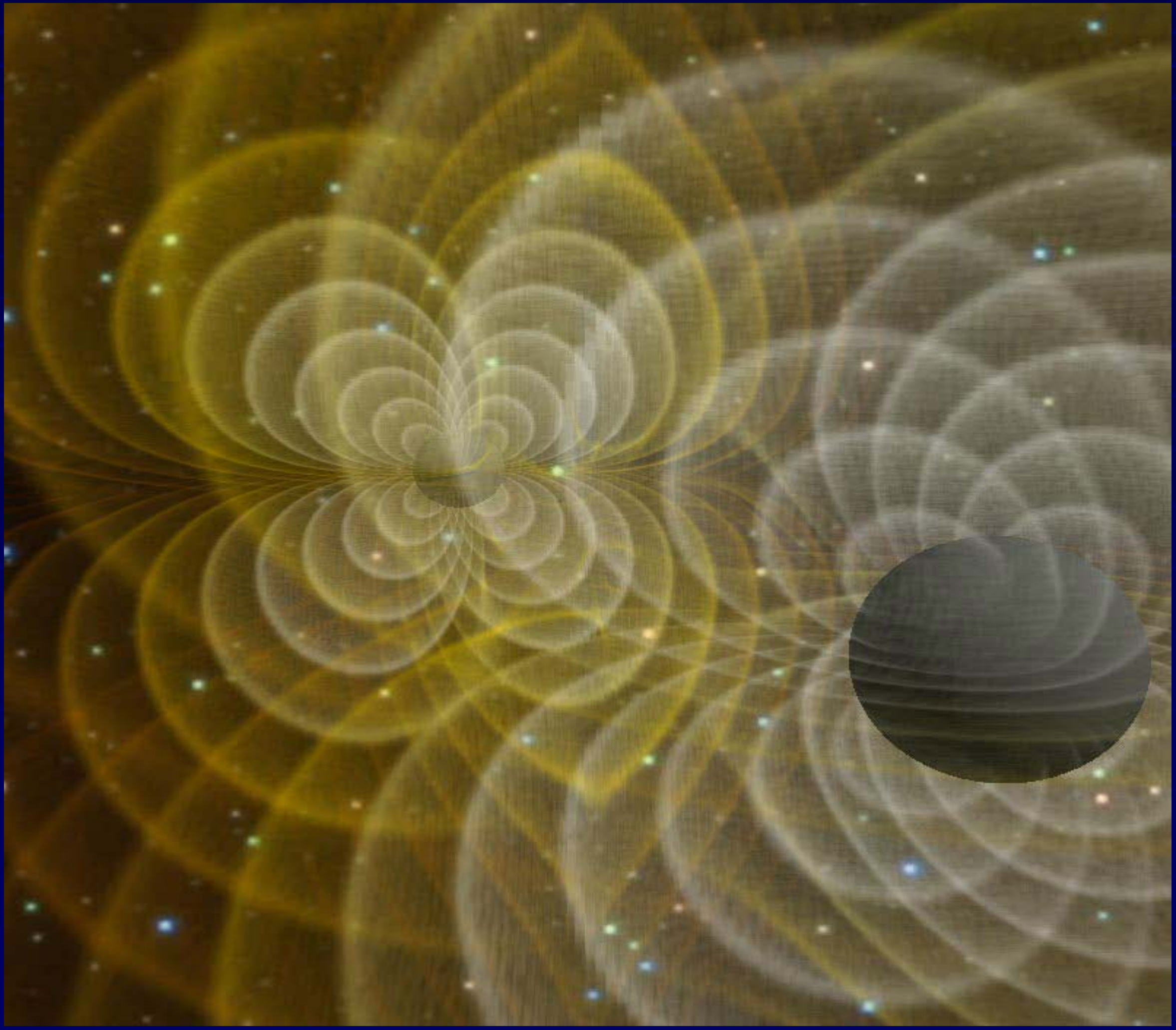
Moving punctures work



Methods

- Usual conformal decomposition: $g_{ij}=\psi^4\gamma_{ij}=e^{4\phi}\gamma_{ij}$
- Initial Bowen-York puncture data: $\psi=1+0.5m_1/r_1+0.5m_2/r_2+u$
- BSSN evolution: γ_{ij}, ϕ, Γ^i, A_{ij}, K
- Gauge: 1+log slicing, Gamma-driver shift allowed to be non-vanishing at the puncture
- 4th order Runge-Kutta, 4th order spatial differencing*, 5th order interpolation at interfaces.
- Adaptive mesh refinement: finest around bh's.

Gauge

$$\partial_t \alpha = -2\alpha K + \beta^j \partial_j \alpha$$

$$\partial_t \beta^i = \frac{3}{4}\tilde{\Gamma}^i + \beta^j \partial_j \beta^i - \eta \beta^i$$

Note that if $\eta = 0$, the homogeneous equation,

$$\partial_t \beta^x - \beta^x \partial_x \beta^x = 0$$

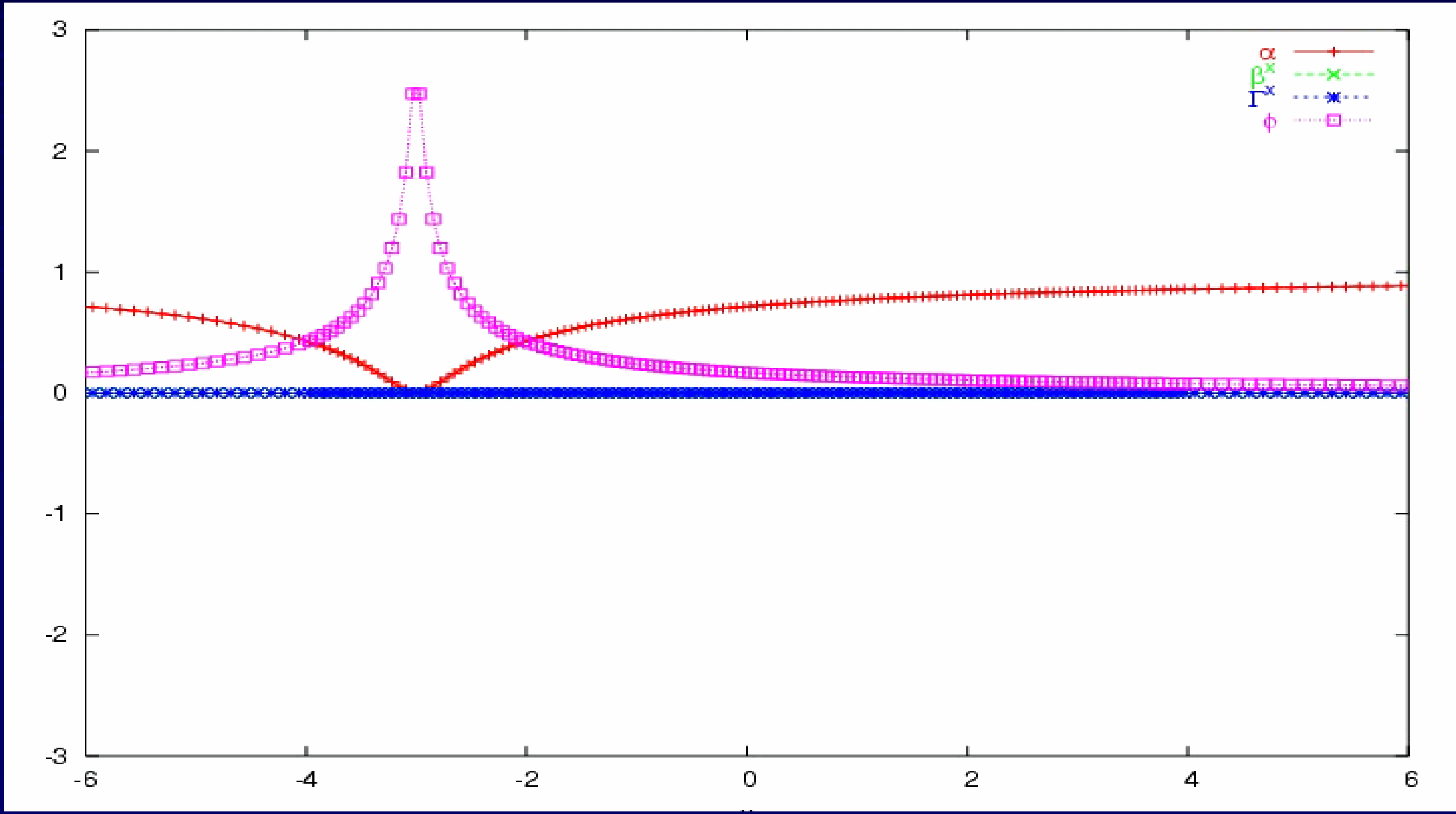
admits the shock wave solution,

$$\beta^x = \frac{x - x_0}{t_0 - t}$$

However, the source term $\frac{3}{4}\tilde{\Gamma}^i$ is driven by derivatives of β^i:

$$\partial_t \tilde{\Gamma}^i = \ldots + \tilde{\gamma}^{jk}\partial_j \partial_k \beta^i + \frac{1}{3}\tilde{\gamma}^{ij}\partial_j \partial_k \beta^k$$
$$-\tilde{\Gamma}^j \partial_j \beta^i + \frac{2}{3}\tilde{\Gamma}^i \partial_j \beta^j + \ldots$$

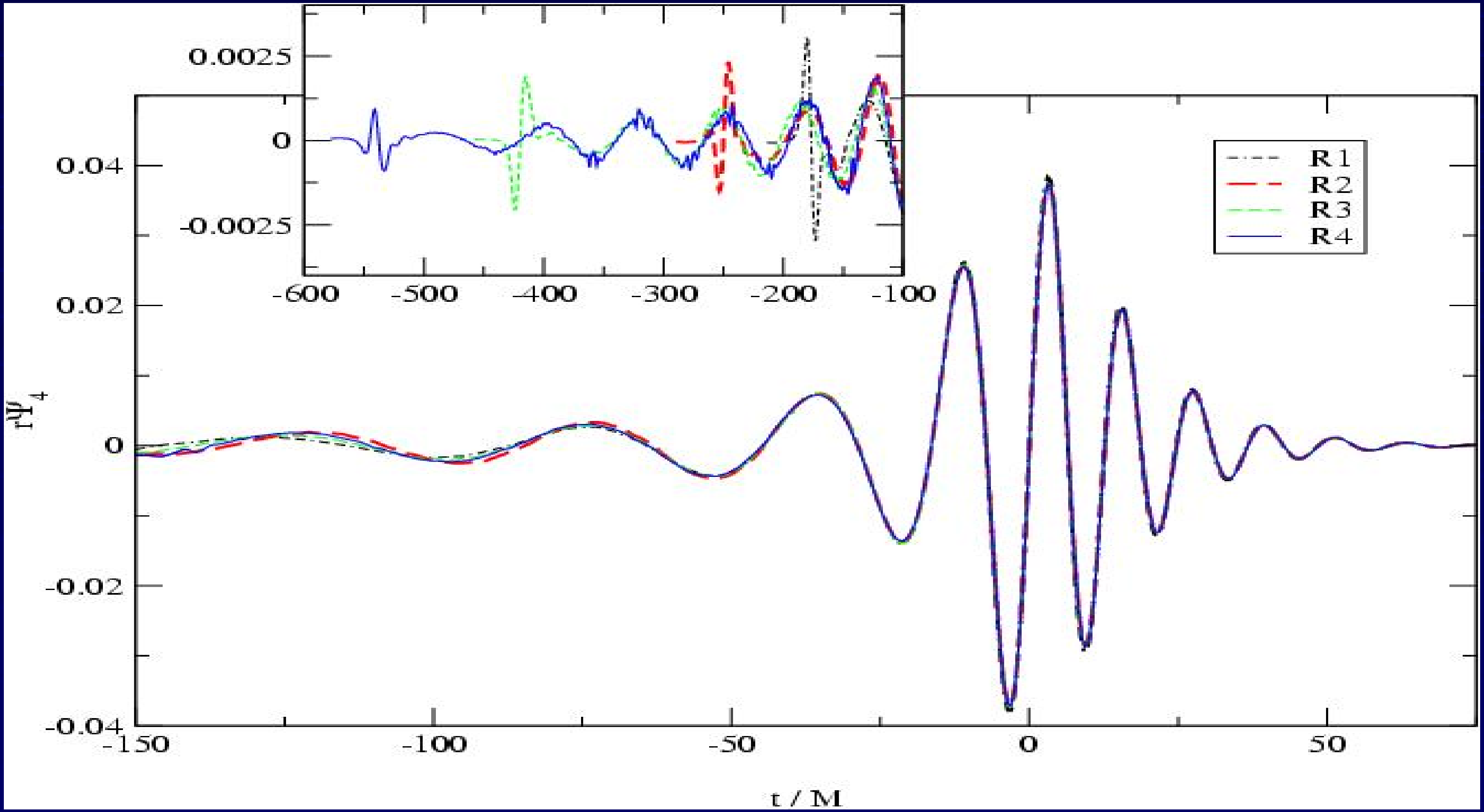
Gauge test



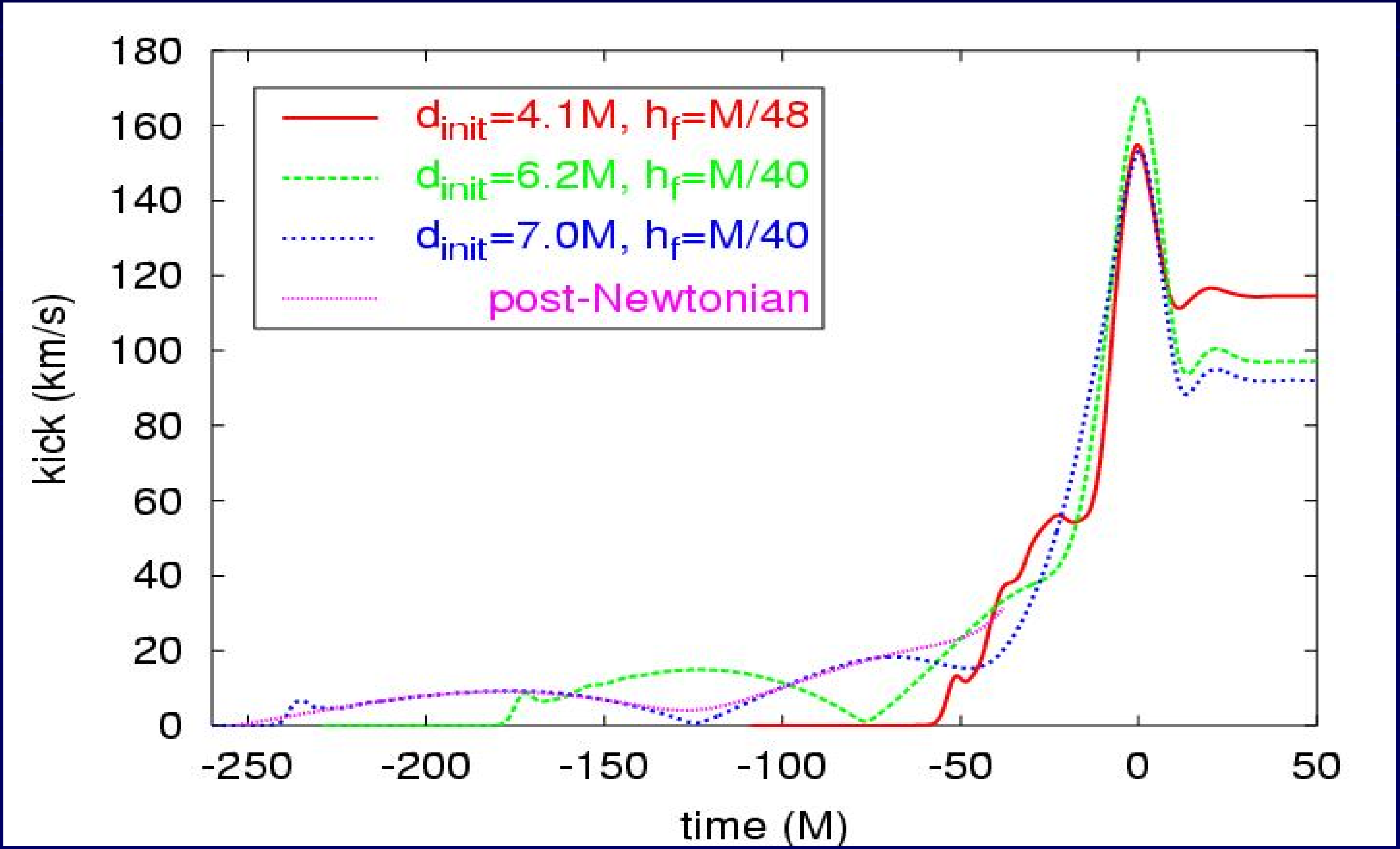
Applications

- Equal-mass, various separations
  - Baker et al, Bruegmann et al, Campanelli et al
- Non-equal mass, recoil computation
  - Baker et al, Gonzalez et al, Herrmann et al
- Spins, aligned and anti-aligned
  - Campanelli et al

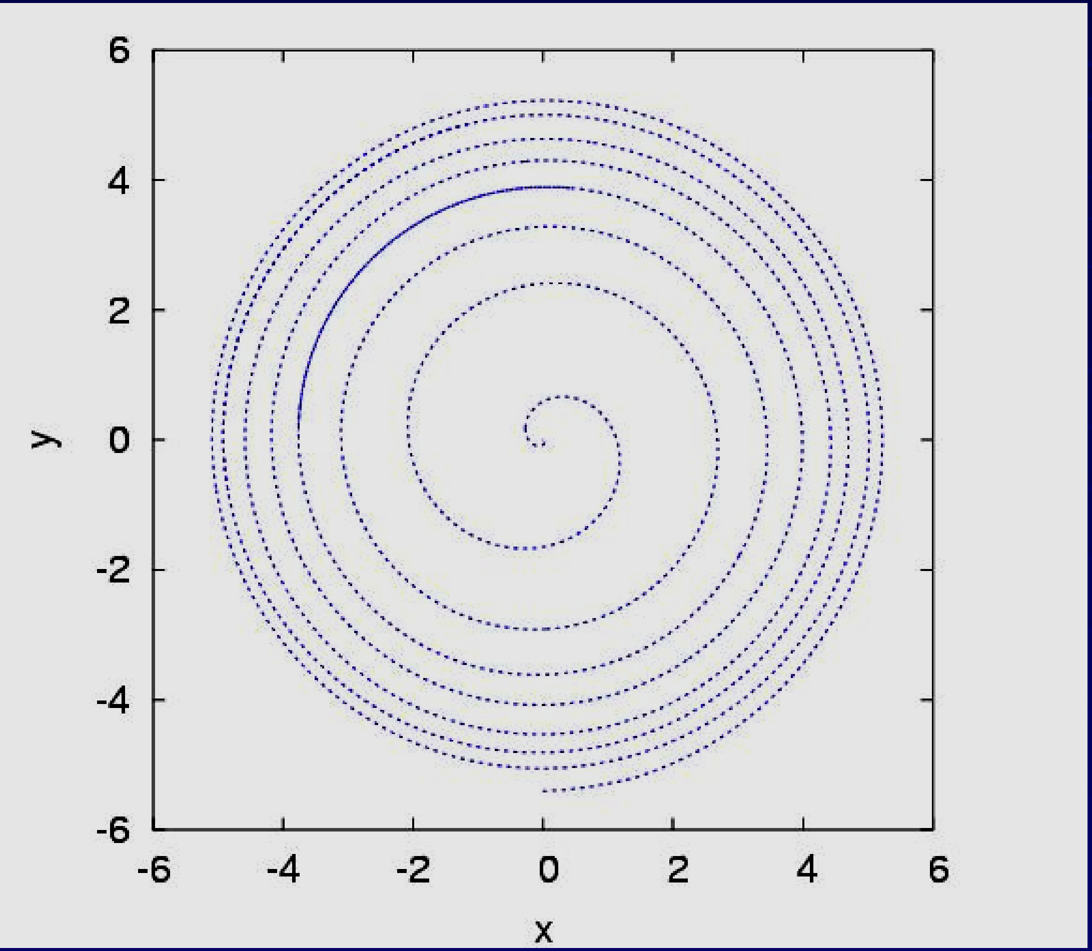
Equal-mass, various separations



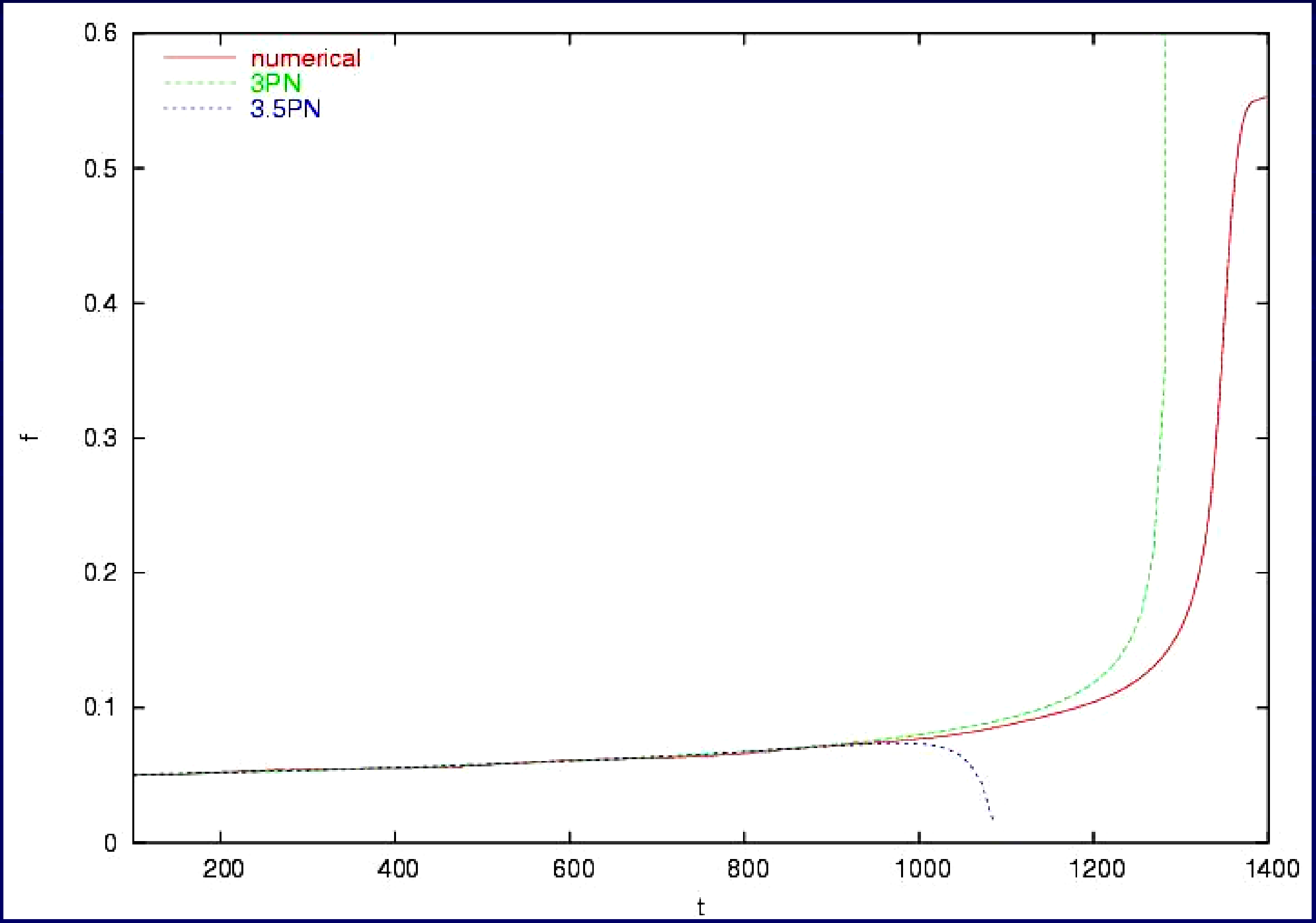
Non-equal-mass, radiative recoil



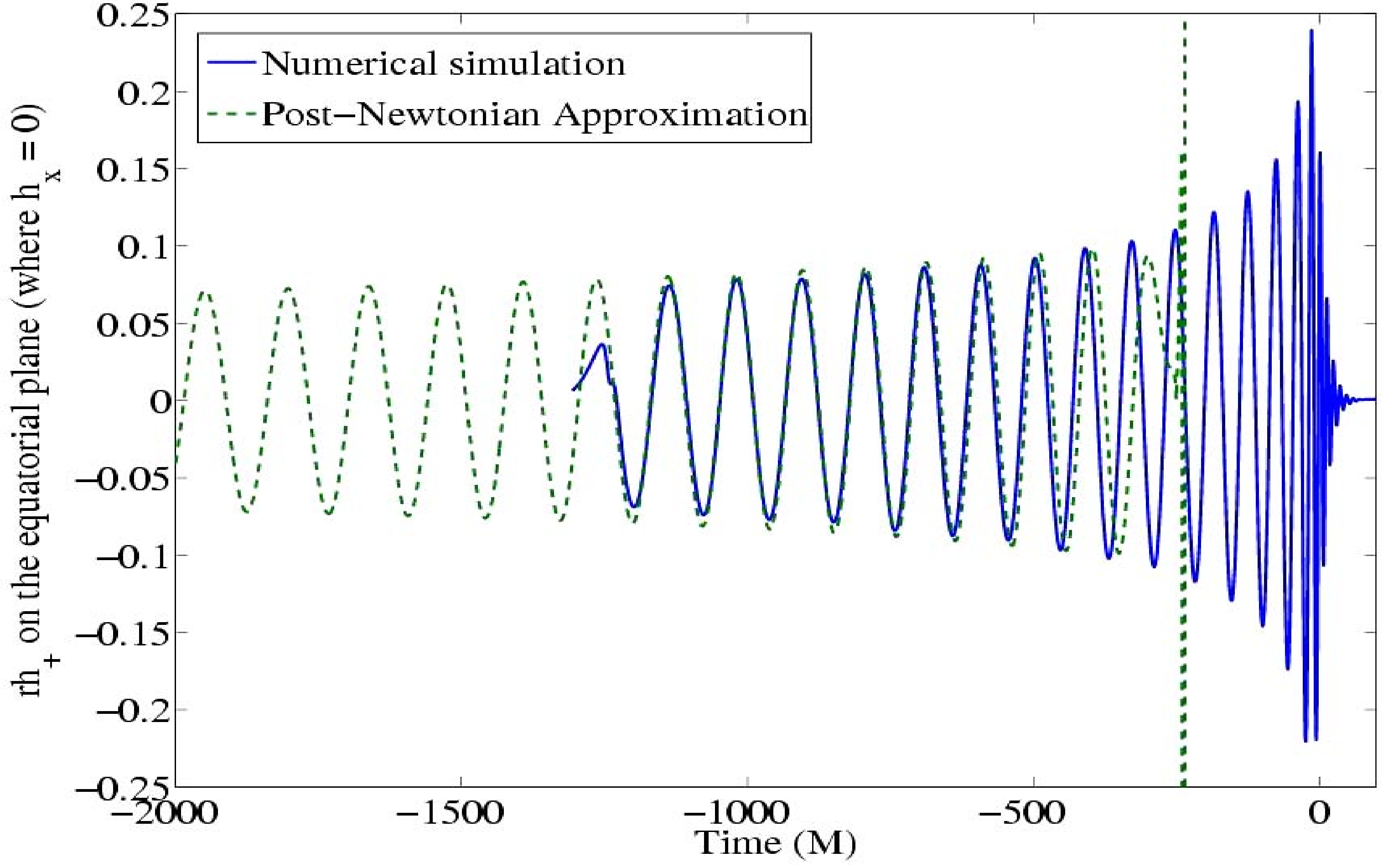
Equal mass, 8-orbit run



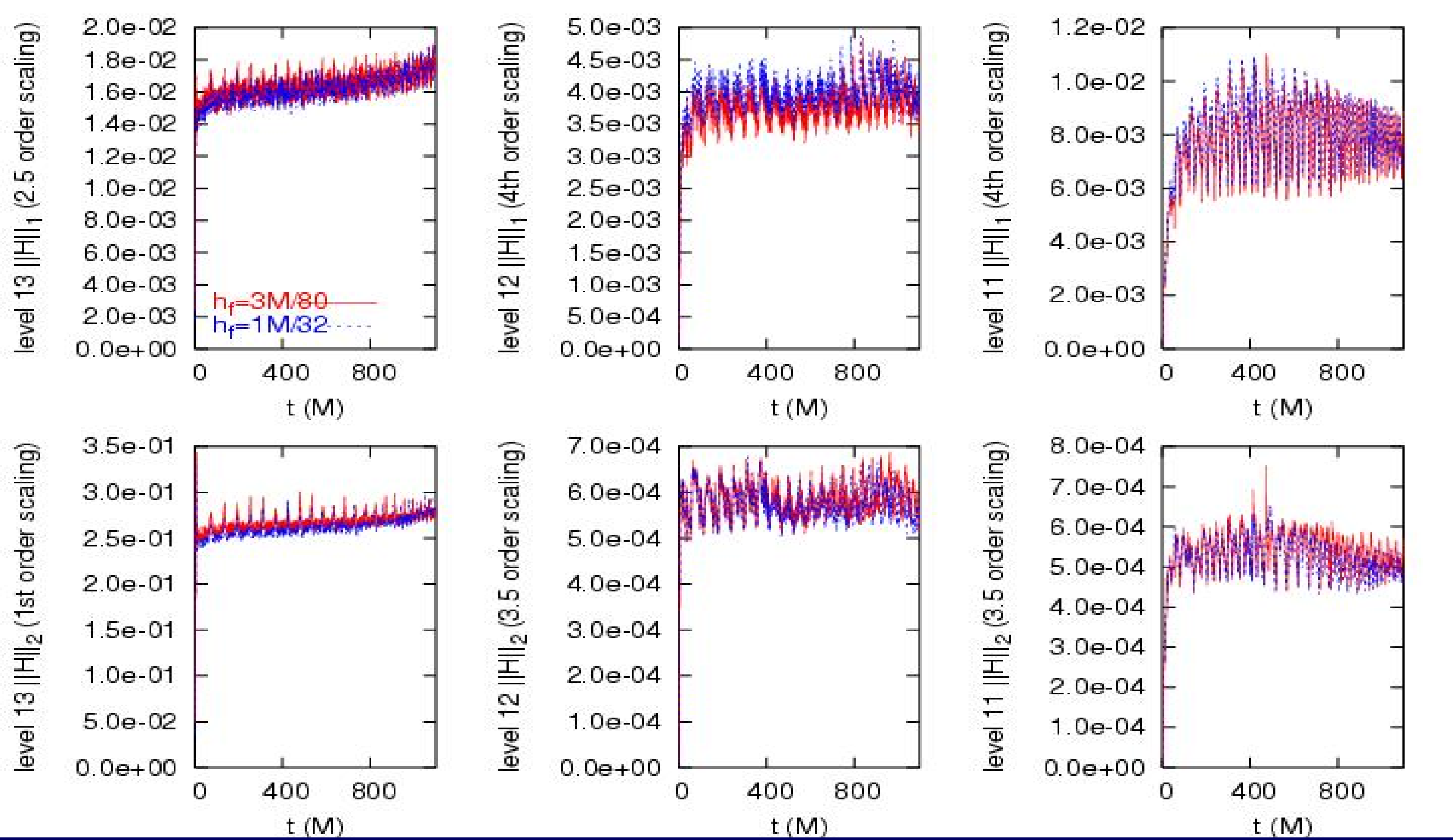
GW frequency compared with PN



h+ compared with PN



Hamiltonian convergence



Note on L1 and L2 norms

$$||\text{error}||_1 \sim \frac{N^0 h^p + N^2 h^r + N^3 h^b}{N^3}$$

$$||\text{error}||_2 \sim \sqrt{\frac{N^0 h^{2p} + N^2 h^{2r} + N^3 h^{2b}}{N^3}}$$

$$N \sim h^{-1}$$

$p =$ order of puncture error

$r =$ order of refinement boundary error

$b =$ order of bulk error

Away from the puncture, if $b = 4$ and $r = 3$ then

$$||H||_1 \sim h^4$$

$$||H||_2 \sim h^{3.5}$$

But near the puncture, if $p < 1$, then

$$||H||_1 \sim h^{p+3}$$

$$||H||_2 \sim h^{\frac{2p+3}{2}}$$

$$\psi \sim |\vec{x} - \vec{x}_{\text{punc}}|^{-\frac{1}{2}} \rightarrow p = -\frac{1}{2}\ ?$$

Notes on puncture differencing

- So far, punctures have been confined to a plane between grid points.
- If error doesn't propagate from the puncture (e.g. via gauge mode) then it's not a concern.
- For differencing error within distance of $O(h)$ from the puncture to be $O(h^4)$, fields must be C^4.
- Regarding the conformal factor variable:

If $e^{\phi} \sim |\vec{x} - \vec{x}_{\rm punc}|^{-\frac{1}{2}}$ then for positive, even n, $\chi \equiv e^{-n\phi}$ is $C^{\frac{n}{2}-1}$ if n is not divisible by 4 and C^{∞} if n is divisible by 4.

But note terms in the evolution equations of the form

$$e^{-4\phi}\partial_i\partial_j\phi = \frac{1}{n}\chi^{\frac{4}{n}}\left(\partial_i\chi\partial_j\chi/\chi^2 - \partial_i\partial_j\chi/\chi\right)$$

A modified BSSN formulation

Assuming $\lim_{\vec{x}\to\vec{x}_{punc}} \alpha = O(|\vec{x} - \vec{x}_{punc}|^2)$, define:

$$
\begin{aligned}
w &= e^{-2\phi} \\
q &= e^{2\phi}\alpha
\end{aligned}
$$

Then the BSSN equations become:

$$
\begin{aligned}
\partial_t w &= \frac{1}{3} q w^2 K - \frac{1}{3} w \partial_k \beta^k + \beta^k \partial_k w \\
\partial_0 K &= -\tilde{\gamma}^{ji} w^2 \nabla_j \nabla_i (qw) + qw \left(\tilde{A}_{ab}\tilde{A}^{ab} + \frac{1}{3} K^2 \right) \\
\partial_0 \tilde{\gamma}_{ij} &= -2qw\tilde{A}_{ij} \\
\partial_0 \tilde{A}_{ij} &= w \left[-w \nabla_i \nabla_j (qw) + q w^2 R_{ij} \right]^{\mathrm{TF}} \\
&\quad + qw \left(K\tilde{A}_{ij} - 2\tilde{A}_{ia}\tilde{A}^a{}_j \right) \\
\partial_t \tilde{\Gamma}^i &= 2q \left(w \tilde{\Gamma}^i_{ab}\tilde{A}^{ab} - \frac{2}{3} w \tilde{\gamma}^{ia} K_{,a} + 6\tilde{A}^{ia} w \phi_{,a} \right) \\
&\quad + \tilde{\gamma}^{kl} \left(-\tilde{\Gamma}^j_{kl}\beta^i{}_{,j} + \frac{2}{3}\tilde{\Gamma}^i_{kl}\beta^j{}_{,j} \right) + \beta^k \tilde{\Gamma}^i{}_{,k} \\
&\quad + \tilde{\gamma}^{jk}\beta^i{}_{,jk} + \frac{1}{3}\tilde{\gamma}^{ij}\beta^k{}_{,kj} - 2\tilde{A}^{ia}(qw)_{,a} \\
\partial_t q &= -2qK - \frac{1}{3} q^2 w K + \frac{1}{3} q \partial_k \beta^k + \beta^k \partial_k q \\
\partial_t \beta^i &= \frac{3}{4}\tilde{\Gamma}^i + \beta^j \partial_j \beta^i - \eta \beta^i
\end{aligned}
$$

where:

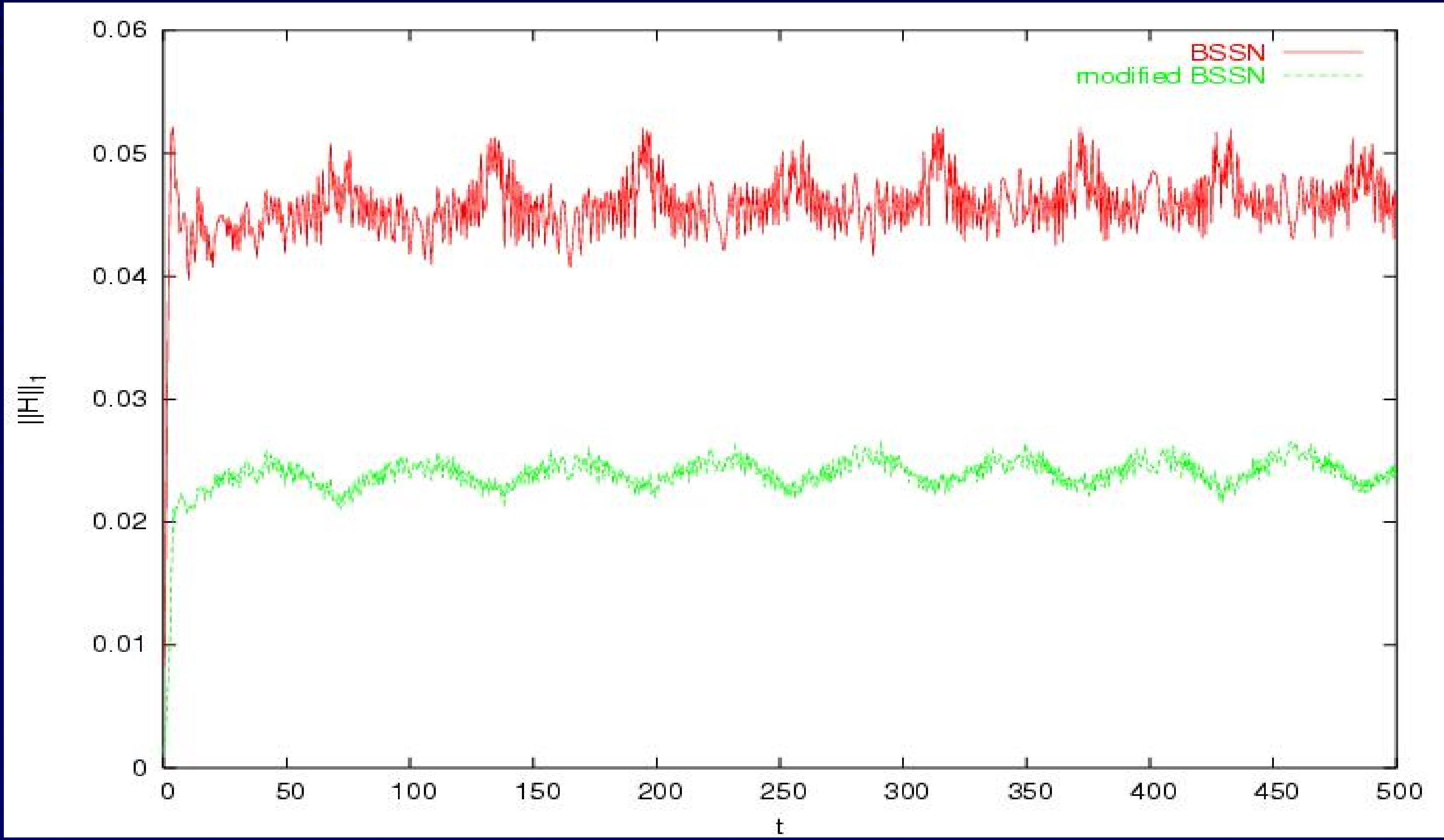
$$
\begin{aligned}
w \nabla_m \nabla_n (qw) &= w \partial_m \partial_n (qw) - 4w \partial_{(m}\phi \partial_{n)}(qw) \\
&\quad - w \tilde{\Gamma}^k_{mn} \partial_k (qw) + 2\tilde{\gamma}_{mn}\tilde{\gamma}^{kl} w \partial_k \phi \partial_l (qw)
\end{aligned}
$$

$$
\begin{aligned}
w^2 R_{ij} &= w^2 \tilde{R}_{ij} - 2w^2 \tilde{\nabla}_i \tilde{\nabla}_j \phi - 2\tilde{\gamma}_{ij} w^2 \tilde{\nabla}^k \tilde{\nabla}_k \phi \\
&\quad + 4w \partial_i \phi w \partial_j \phi - 4\tilde{\gamma}_{ij} w \partial^k \phi w \partial_k \phi
\end{aligned}
$$

$$
w^2 \tilde{\nabla}_i \tilde{\nabla}_j \phi = w^2 \partial_i \partial_j \phi - \tilde{\Gamma}^k_{ij} w^2 \partial_k \phi
$$

$$
\begin{aligned}
w^2 \partial_i \partial_j \phi &= \frac{1}{2} \left(\partial_i w \partial_j w - w \partial_i \partial_j w \right) \\
w \partial_i \phi &= -\frac{1}{2} \partial_i w
\end{aligned}
$$

Modified BSSN test



Recap and open questions

- Moving punctures yield stable and accurate evolutions in a variety of cases.
- Does error ever propagate from punctures?
- What is the best lapse condition?
  - Asymptotic behavior near puncture
  - Gauge speed
- What is the best variant of BSSN?
  - $\chi=\psi^{-n}$?
  - Densitize other variables to make C^4 ?