High order multi-block/patch evolutions of Einstein's equations: an update

or
From differential geometry to multiple blocks
or
From energy estimates to numerical stability

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More details and collaborators:

- "Multi-block simulations in general relativity: High order discretizations, numerical stability, and applications", L. Lehner, O. Reula, M. Tiglio, Class. Quant. Grav 22, 5283 (2005)

- "Boundary conditions for Einstein's field equations: analytical and numerical analysis", O. Sarbach and M. Tiglio, Journal of Hyperbolic Diff. Equations 2, 839 (2005).

- "New, efficient, and accurate high order derivative and dissipation operators satisfying summation by parts, and applications in three-dimensional multi-block evolutions". P.Diener, N. Dorband, E. Schnetter, M. Tiglio, submitted to Journal of Scientific Computing (2005)

- "A Multi-block infrastructure for three-dimensional time-dependent numerical relativity", E. Schnetter, P. Diener, N. Dorband, M. Tiglio, Class.Quant.Grav.23, S553 (2006)

- "A Numerical study of the quasinormal mode excitation of Kerr black holes", N. Dorband, E. Berti, P. Diener, E. Schnetter, M. Tiglio, Phys.Rev. D74, 084028 (2006)

- "Gravitational wave extraction from numerical spacetimes using a generalized black hole perturbation formalism", N. Dorband, E. Pazos, C. Palenzuela, E. Schnetter, M. Tiglio, in preparation.

- "Solving the Einstein constraints in unstructured and semi-structured grids", O. Korobkin, B. Aksoylu, M. Holst, M. Tiglio, work in progress.

Overview

- Multiple blocks

- Numerical stability

- High order methods

- Efficiency and accuracy

- 3D simulations:

    – Scalar perturbations of a Kerr black hole
    – Formulation of the Einstein equations
    – Solving the constraints with finite elements.
    – Wave extraction techniques
    – Weak gravitational waves
    – Distorted black holes

Multiple blocks

In the same way in differential geometry one covers the manifold with several patches, here we cover the computational domain with several blocks (non-overlapping patches)
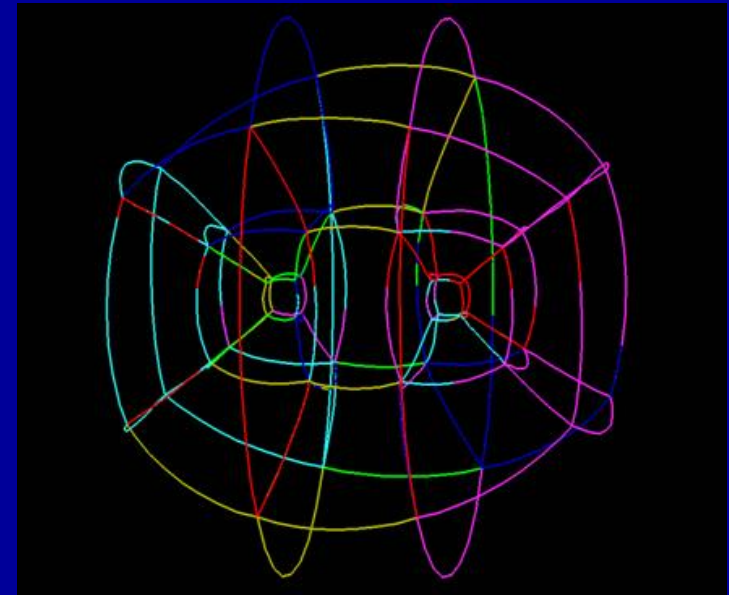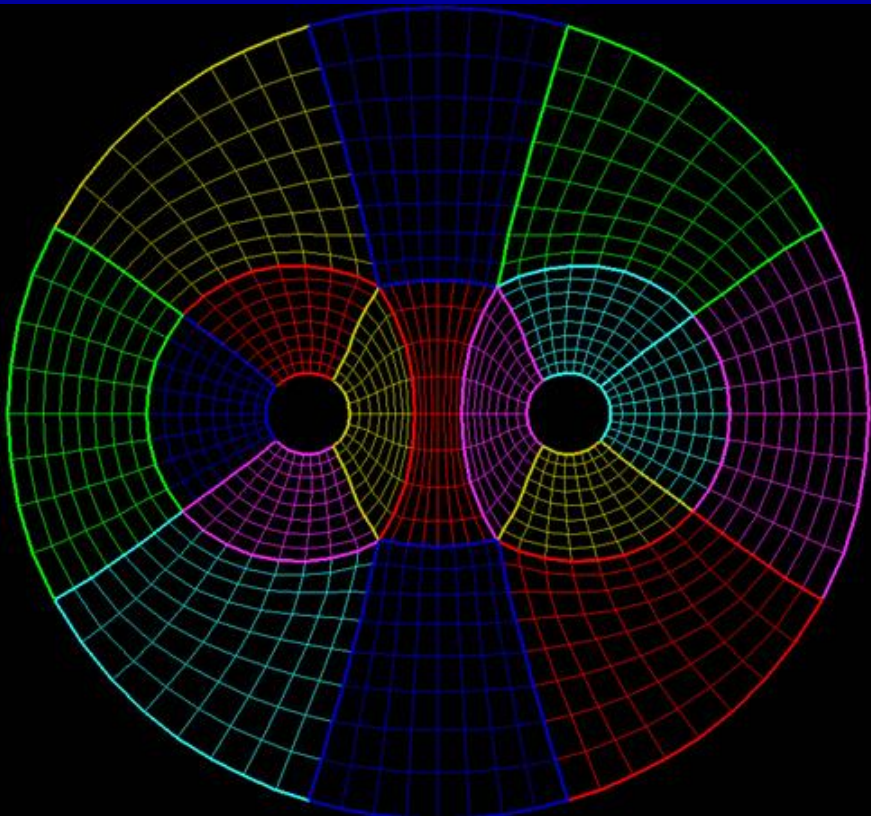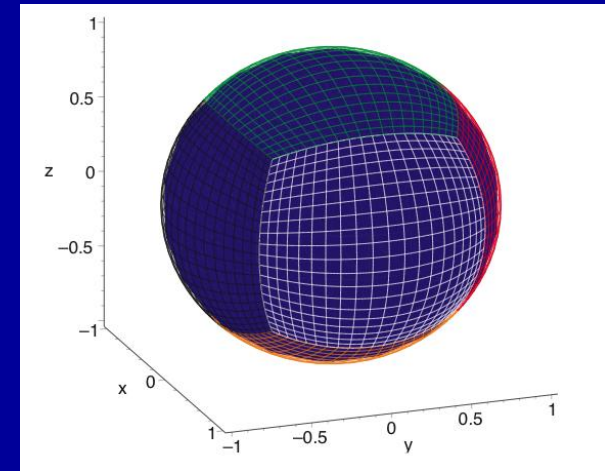
Multiple blocks
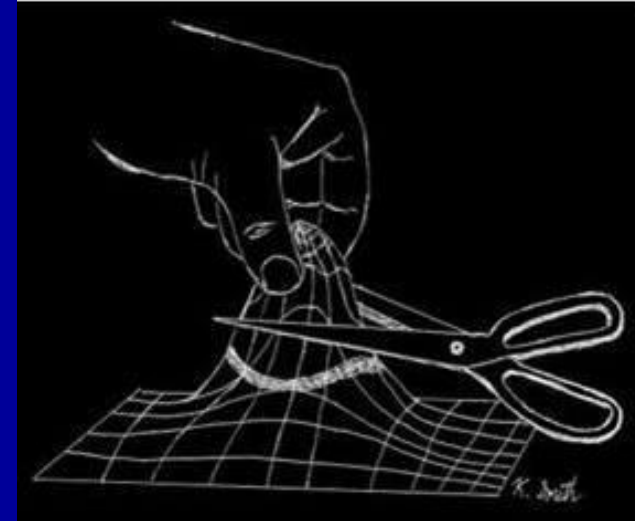
- **Advantages:**

  1) We can handle non-trivial topologies
  2) Smooth (inner and outer boundaries)
  3) One keeps the angular resolution fixed.
  4) With grid generation software, one can handle "arbitrary" complicated geometries.

- **Some applications:**

  1) Closed cosmologies
  2) Well defined black hole excision and outer boundary conditions. No need to extend the eqs beyond null infinity if one compactifies the spacetime
  3) Moving the boundaries far away becomes an order N problem (as opposed to N^3 with cartesian coordinates)

Numerical stability

For symmetric hyperbolic systems with maximally dissipative boundary conditions, one can show well posedness of the associated initial-boundary value problem through the energy method

An energy estimate is derived, using that the system is symmetric and integrating by parts.

Numerical stability is the discrete version of well posedness. Numerical schemes can be built so that these two concepts go hand by hand.

Linearly stable symmetric schemes can be constructed by using difference operators satisfying the discrete version of integration by parts: Summation By Parts (SBP)

In the case of multiple blocks, the different domains are communicated through penalty terms which do not spoil the energy estimate.

Numerical stability

A difference operator D is said to satisfy SBP if

$$\langle u, Dv \rangle + \langle v, Du \rangle = \left. (uv) \right|_a^b$$

For some scalar product $\langle u, v \rangle = h \sum_{i,j=1}^{n} u_i v_j \sigma_{ij}$

The difference operators do not depend on the equation one solves (they need to be derived only once)

In the interior D is a standard centered difference operator of order 2n

If the scalar product is diagonal (no-diagonal), the order at and close to boundaries is n (2n-1)

We denote them by the order in the interior followed by the order at and close to boundaries.

For example: 2-1, 4-2, 4-3, 6-3, 6-5, 8-4, 10-5 (the red ones are not unique)

We can make use of this non-uniqueness to optimize them.

High order methods

- Einstein's equations can be written as a system of first order symmetric hyperbolic equations which are not genuinely non-linear.

- As a result, the solutions are expected to be smooth (no shocks).

- High order and spectral methods are ideal for systems with smooth solutions.

- They are especially useful in long term evolutions, low order schemes tend to have large phase errors.

Efficiency and accuracy

- <u>Efficiency:</u> By construction the principal part of the semidiscrete equations has purely imaginary eigenvalues. The largest one determines the maximum timestep allowed (CFL limit).

- <u>Accuracy:</u> For a given order, the operators coincide in the interior and are different near (inter-block) boundaries.

- By exploiting this non-uniqueness we minimize an average of the spectral radius and the boundary trucation error. We can minimize the latter by around <u>two</u> orders of magnitude and keep the spectral radius comparable to that one of a low order scheme.

Efficiency and accuracy

III. OPTIMIZATION CRITERIA

We start by fixing some notation. If the accuracy of the difference operator D in the interior is $2p$, then there are b points at and near the boundaries where the order of D is only q. In the diagonal case one has $q = p$, and in the restricted full case it is $q = 2p - 1$. We call b the boundary width. The difference operator at these b points uses (up to) s points to compute the derivative. We call s the boundary stencil size.

When expanding D in a Taylor series one has

$$Du|_{x_i} = \left.\frac{du}{dx}\right|_{x_i} + c_i h^q \left.\frac{d^{q+1}u}{dx^{q+1}}\right|_{x_i} \qquad \text{for } i = 1 \ldots b \quad (11)$$

where h is the grid spacing and $x_i = ih$. We call c_i the *error coefficients*.

In what follows, we consider three cases for each family of operators of a given order, denoted by:

- *Minimum bandwidth:* If there are n free parameters, it is always possible to set n of the derivative coefficients to zero, thereby minimizing the computational cost of evaluating the derivatives in the boundary region.

- *Minimum spectral radius:* In this case, we calculate numerically the eigenvalues of the amplification matrix for a test problem, and choose the parameters to minimize the largest eigenvalue.

- *Minimum ABTE:* We minimize the average boundary truncation error (ABTE), which we define as

$$\text{ABTE} := \left(\frac{1}{b} \sum_{i=1}^{b} c_i^2 \right)^{1/2}. \quad (12)$$

Table VII: Properties of the restricted full norm D_{6-5} operators.

Operator	Minimum bandwidth	Minimum spectral radius	Minimum ABTE
Spectral radius	2.940	1.458	3.194
ABTE	0.0986	0.5380	0.0648
c_1	0.1667	1.3692	−0.0154
c_2	−0.1558	−0.2682	−0.0507
c_3	0.0672	−0.2118	0.1336
c_4	0.0953	0.0097	0.0532
c_5	−0.0433	0.0702	−0.0733
c_6	0.0141	0.1434	0.0187
c_7	−0.0163	−0.0972	−0.0123

Efficiency and accuracy

We can gain around six orders of magnitude in accuracy compared to low order methods, without the sacrifice of a very small timestep.


Errors at fixed time t=4.4
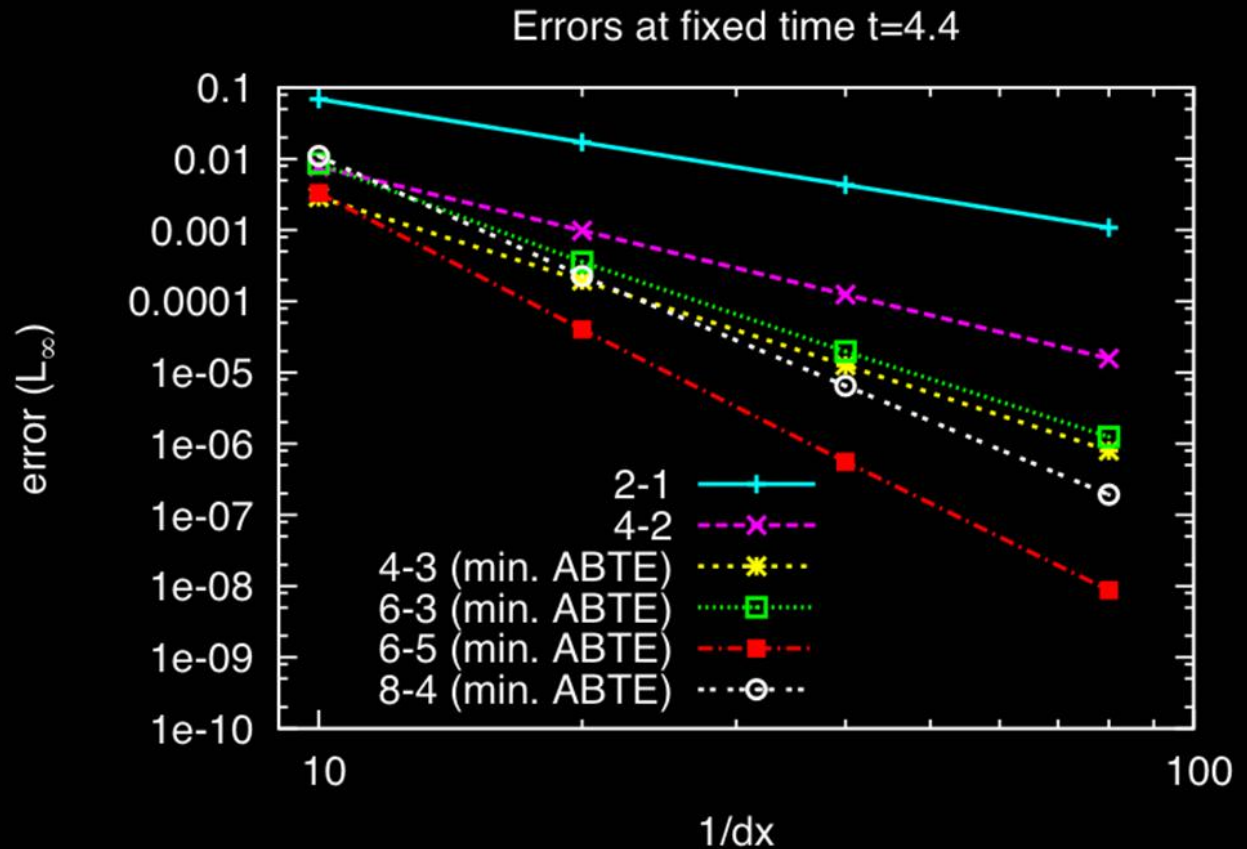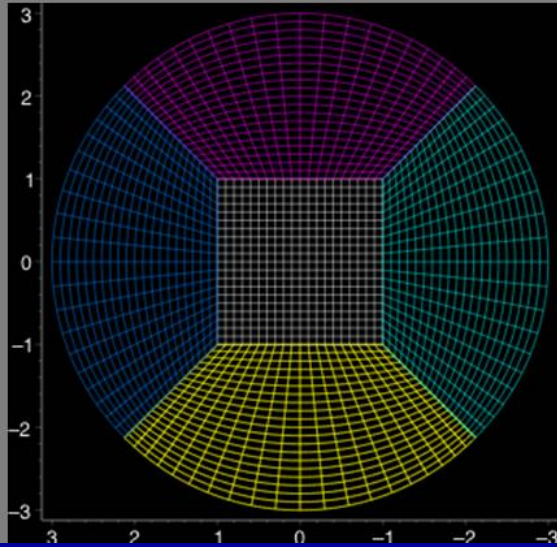
# Scalar perturbations of Kerr black holes

The relative mode excitation of quasinormal modes in ringdown signals
is of special interest to LISA as a test of the no hair conjecture.
Knowledge of the which modes are more likely to get excited would allow to
decrease the number of templates needed for matched filtering

We studied the relative
of different quasinormal
modes and dependence on
the initial data, searching for
initial data which maximally
excites the co- and counter-
rotating fundamental modes
m=2 : σ=2.45
m=-2 : σ=3.43

Overtones become
significantly excited (only) for
large spins (>0.9)
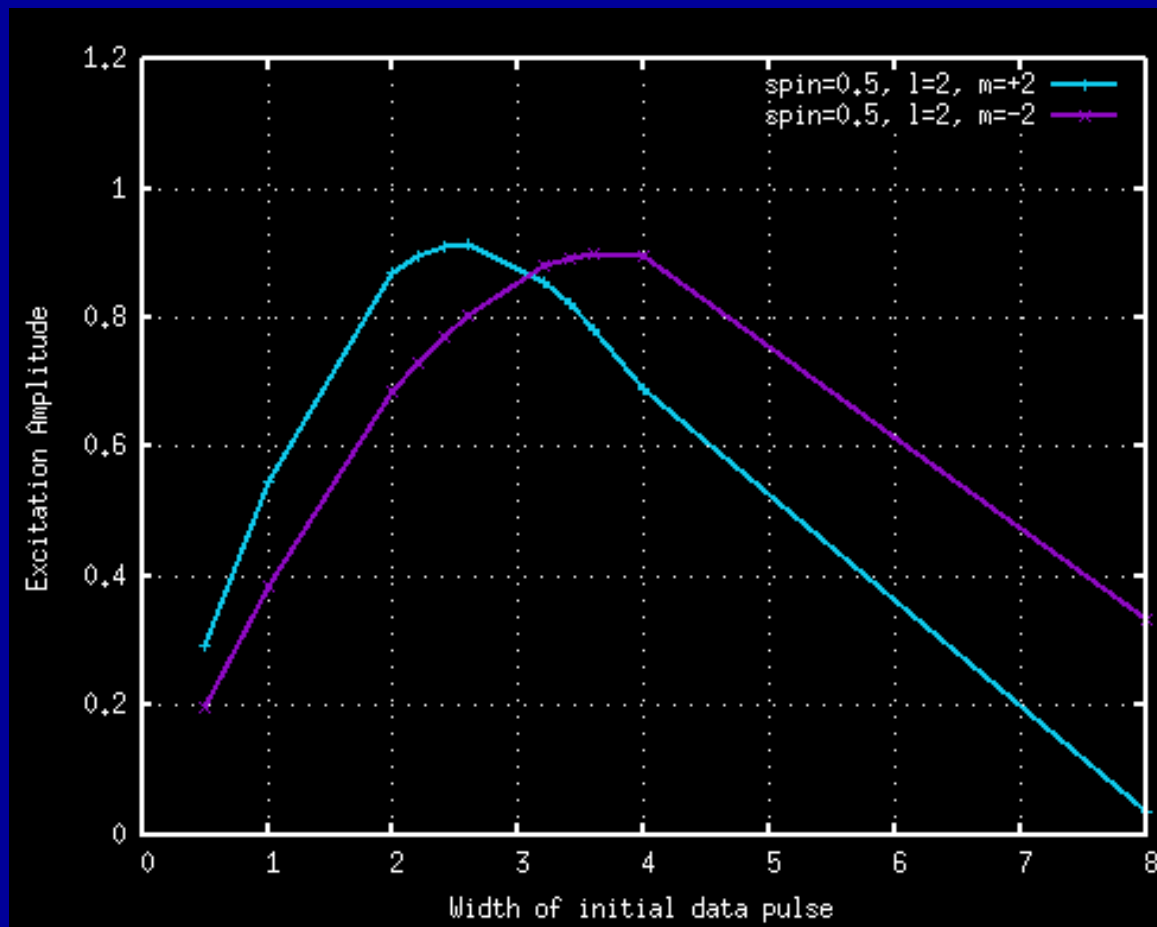
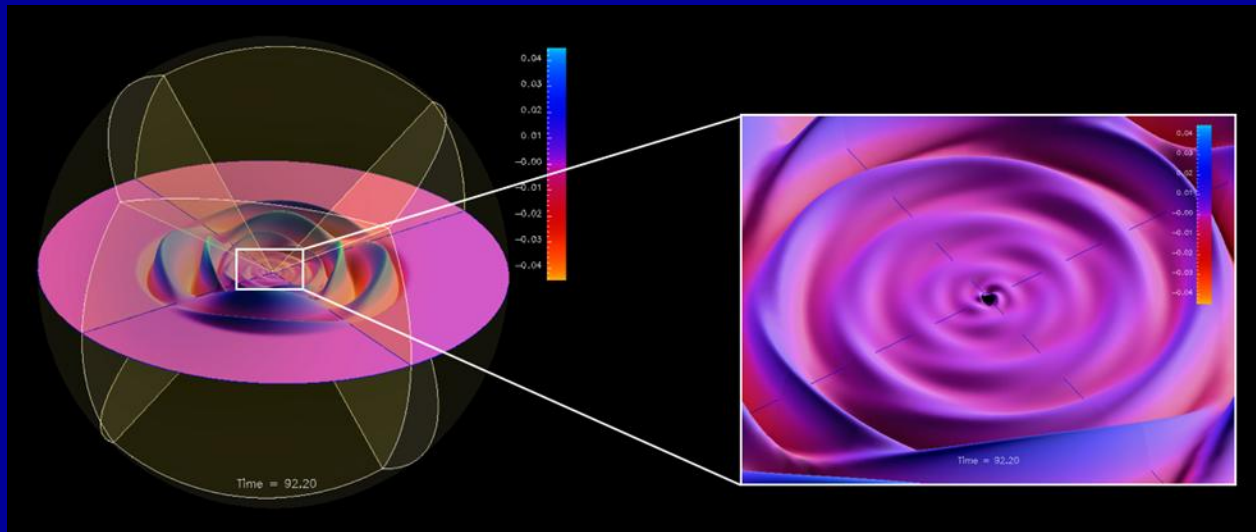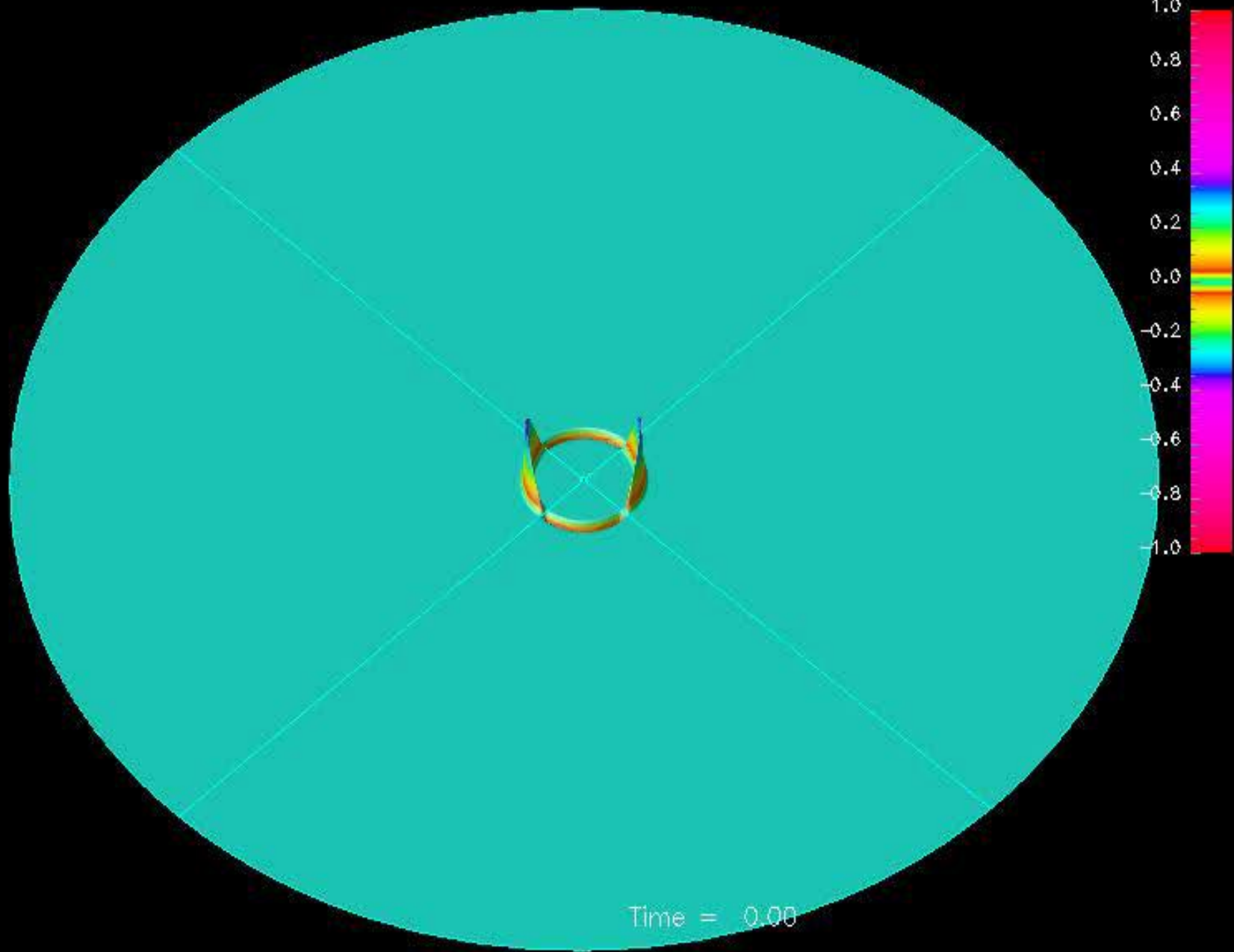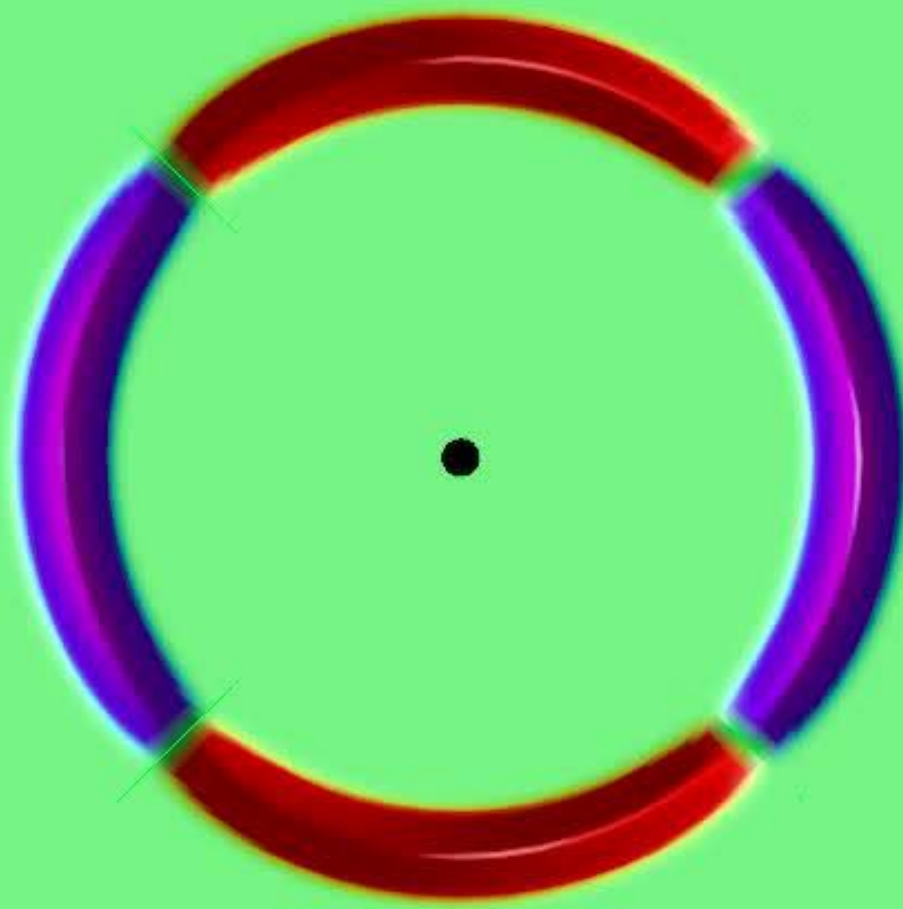# Scalar perturbations of Kerr black holes: tail decay

Tail Run (a=0.5,l=2)
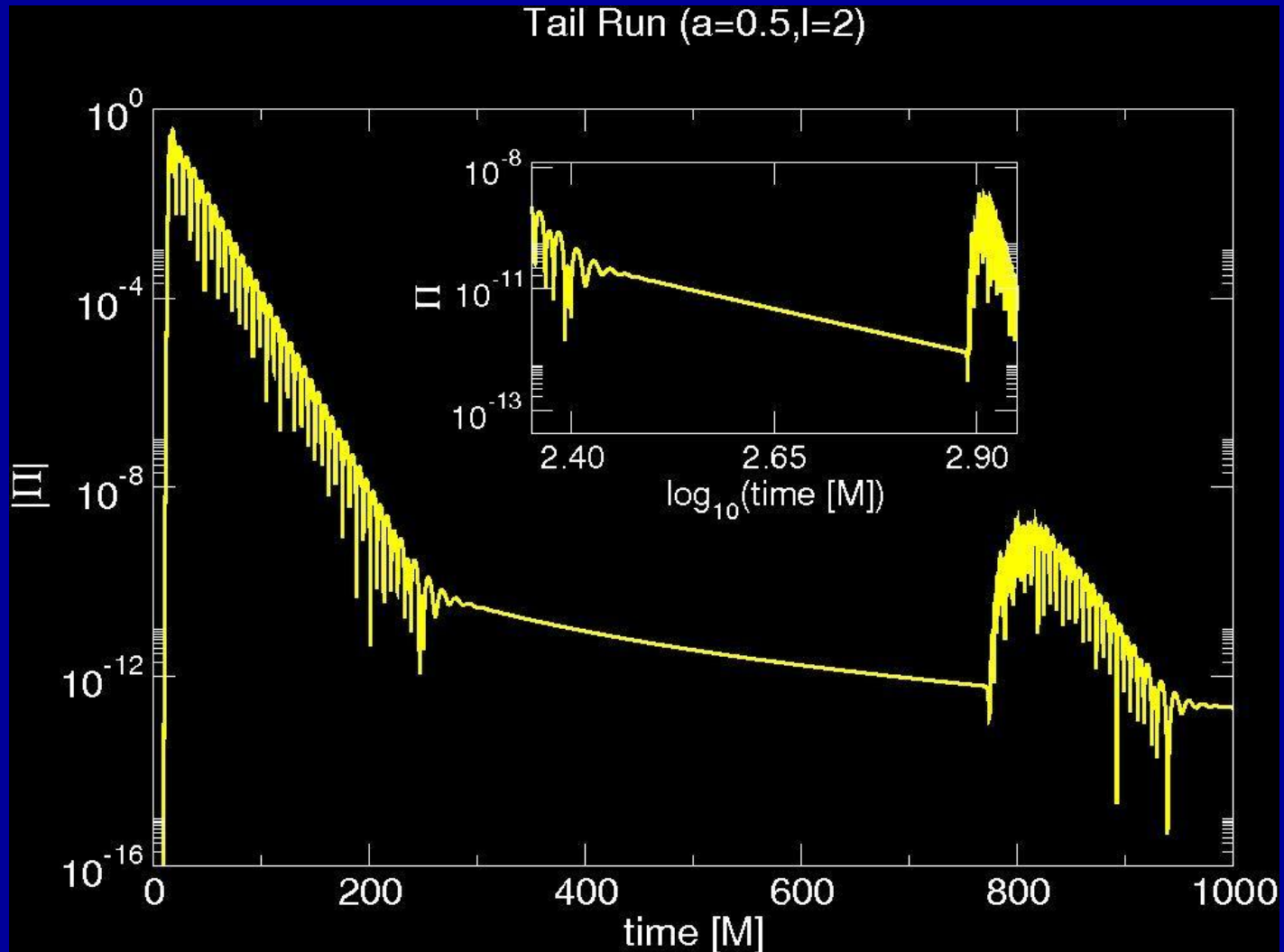
Quasinormal frequencies extracted from the numerical waves.

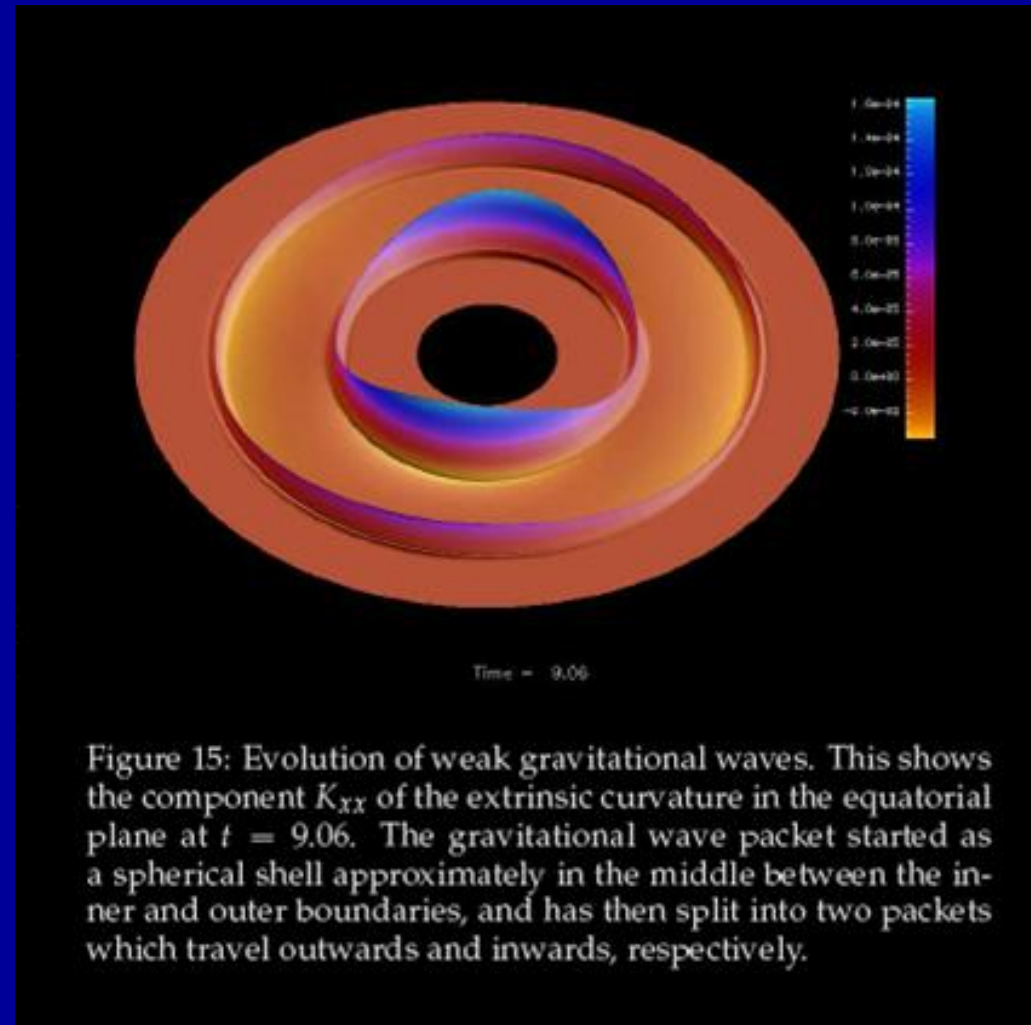| Spin	L=2,m=2	Rel. error	L=2,m=-2	Rel. error
0.0	0.4835 -i*0.09645	$8.66*10^{-5}$ $6.09*10^{-4}$	0.4835 -i*0.09645	$8.66*10^{-5}$ $6.09*10^{-4}$
0.5	0.5859 -i*0.09344	$9.66*10^{-5}$ $5.64*10^{-4}$	0.4228 -i*0.09560	$5.85*10^{-5}$ $1.47*10^{-4}$
0.9	0.7817 -i*0.06929	$1.04*10^{-4}$ $8.61*10^{-4}$	0.3876 -i*0.09371	$5.35*10^{-5}$ $2.47*10^{-4}$

Time = 0.00

Time = 0.20

Formulation of the Einstein equations

- We have two first order symmetric hyperbolic reductions of Einstein's equations

- An Einstein-Christoffel like with Bona-Masso slicing conditions and constraint preserving boundary conditions

- An harmonic formulation of the equations with constraint damping and maximally dissipative boundary conditions (so far)

Time = 9.06

Figure 15: Evolution of weak gravitational waves. This shows the component K_{xx} of the extrinsic curvature in the equatorial plane at $t = 9.06$. The gravitational wave packet started as a spherical shell approximately in the middle between the inner and outer boundaries, and has then split into two packets which travel outwards and inwards, respectively.

Solving the constraints with FeTk(Finite elements ToolKit)

Features of FEtk

Unstructured symplectic meshes

Adaptive 1st order finite elements

Multigrid solver

Parallelization
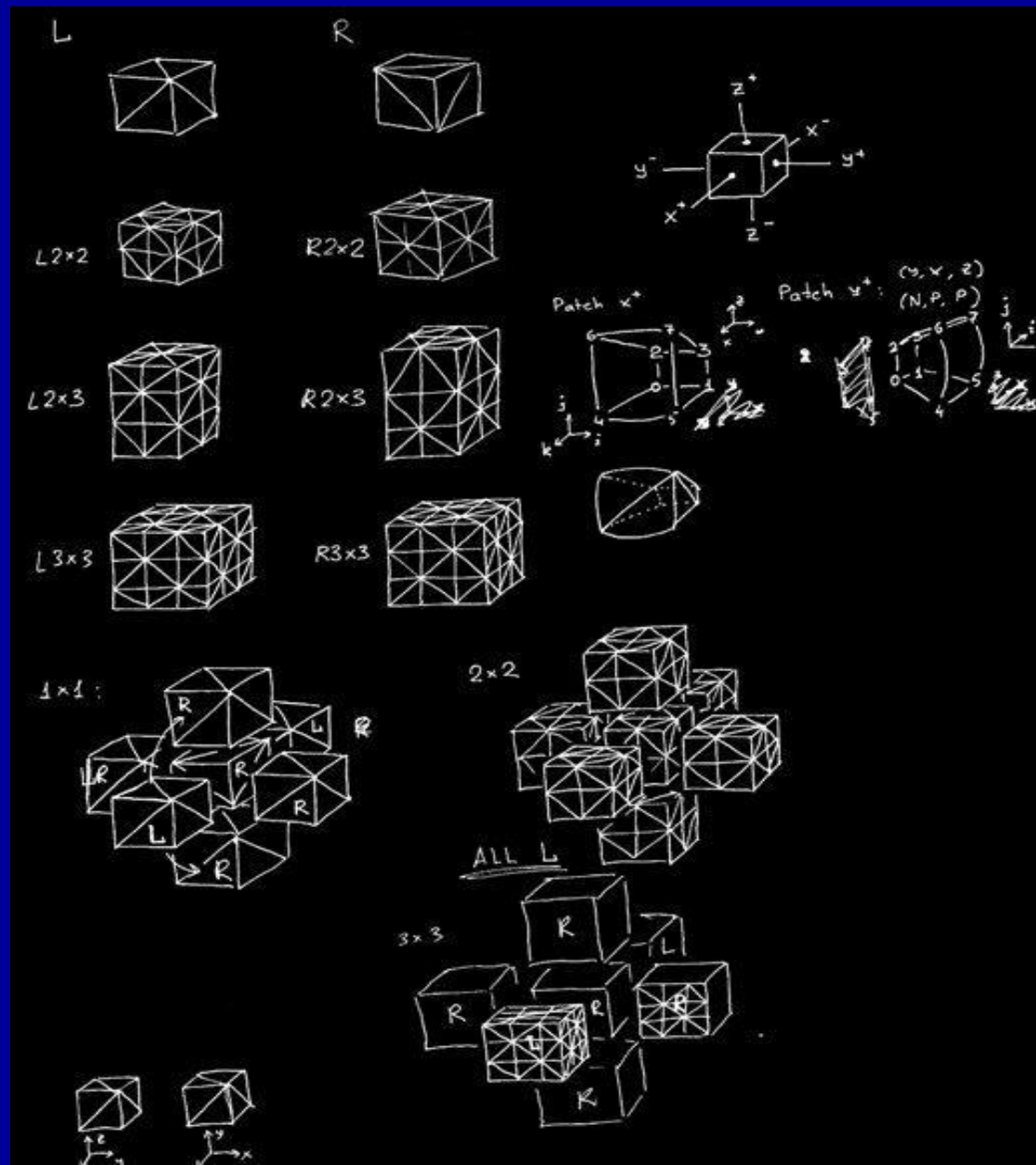
Arbitrary topologies

Weak formulation of the GR constraints
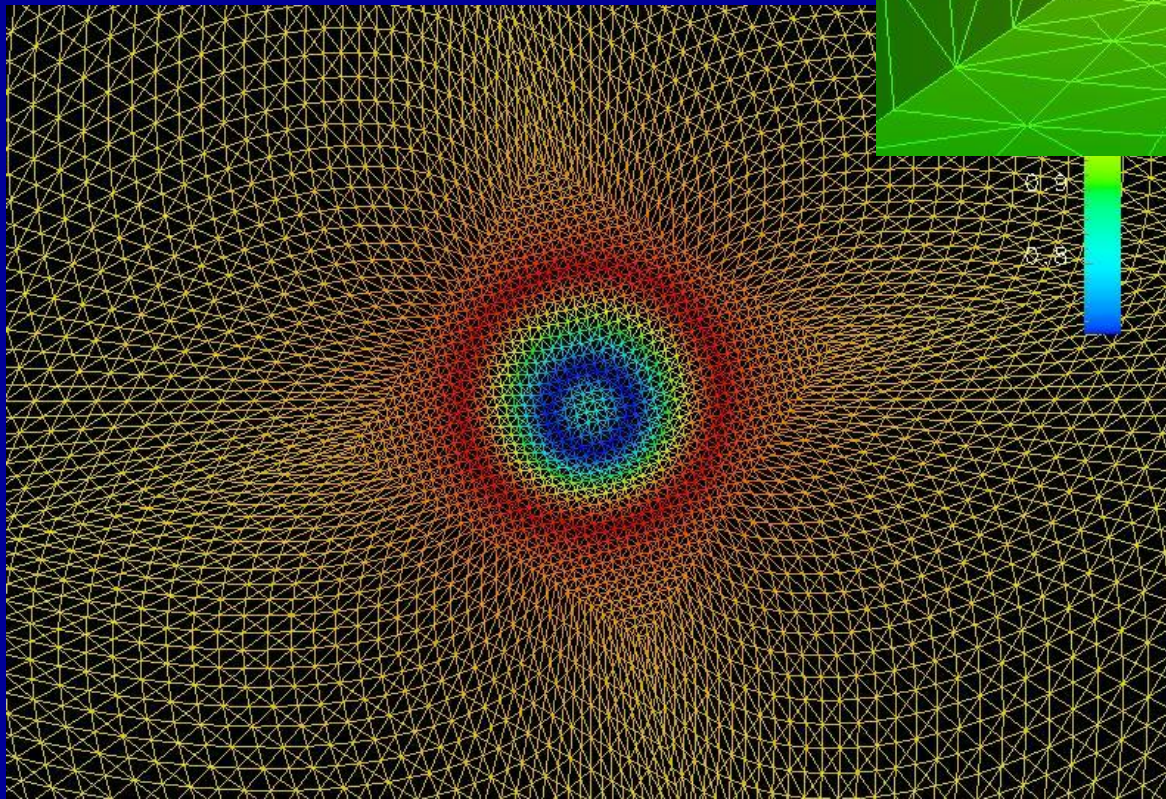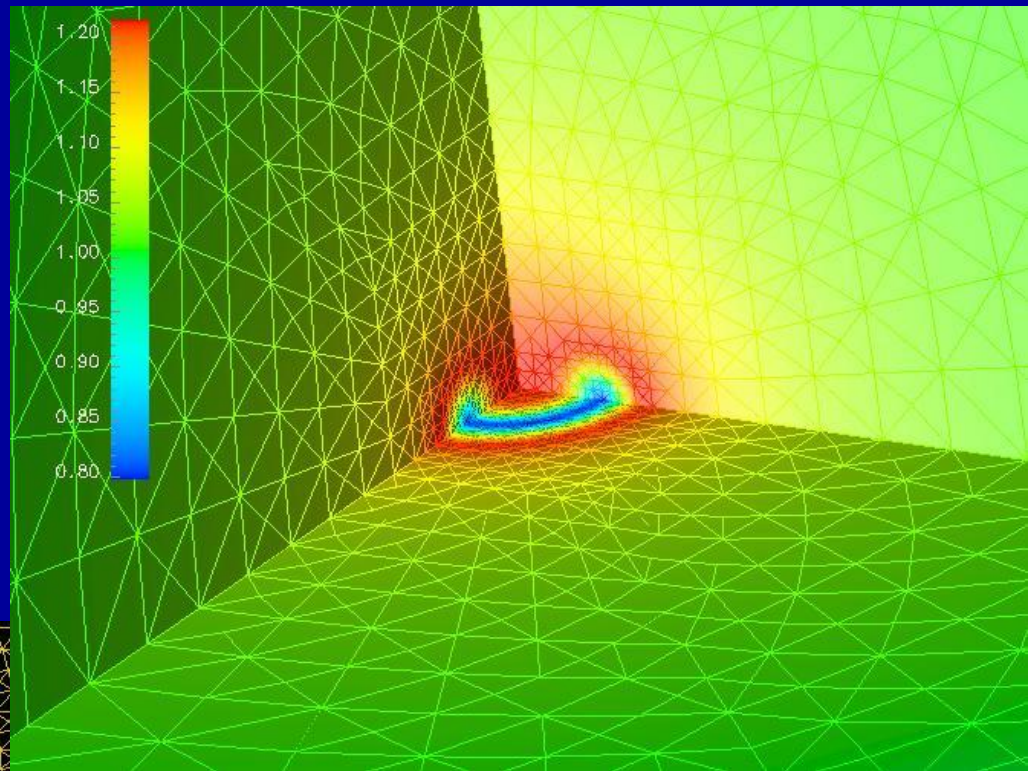
FEtk and Multipatch

Multiblock conformal meshes

No interpolation required to port FeTk solutions

Adaptivity allows to achieve higher accuracy

Brill wave solutions with FeTk

Gravitational wave extraction

- We extract gravitational waves from our numerical spacetime using two different methods: constructing Psi4 and using Regge-Wheeler-Zerilli perturbation theory.

- For this we need to expand Psi4 in spin-weighted spherical harmonics and the metric and the metric in spherical tensorial harmonics.

- In our multi-block grids we always enclose an outer spherical shell, which allows us to compute the integrals needed for these decompositions without the need of interpolation (faster and more accurate).

- We use high order numerical integration over these spheres (same order as evolution scheme itself).
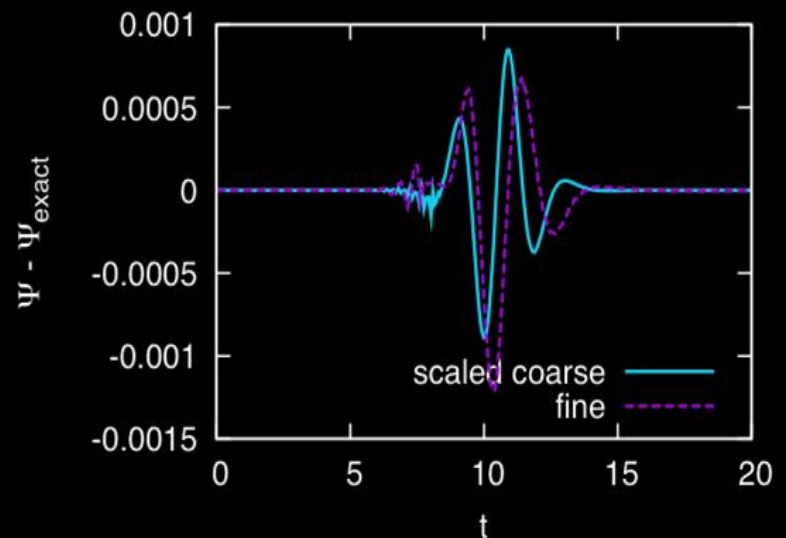
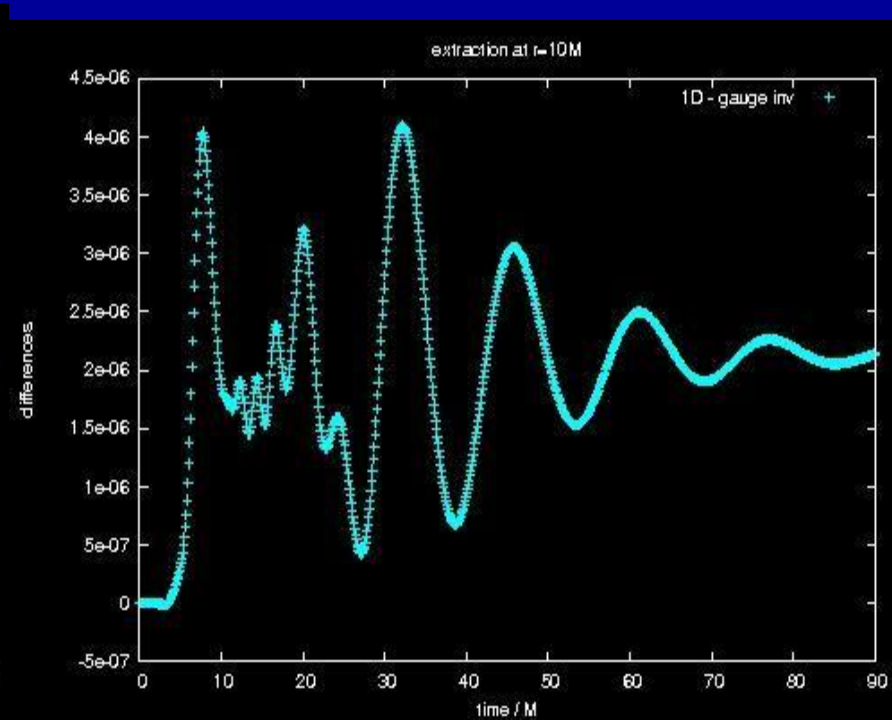Extracted wave form
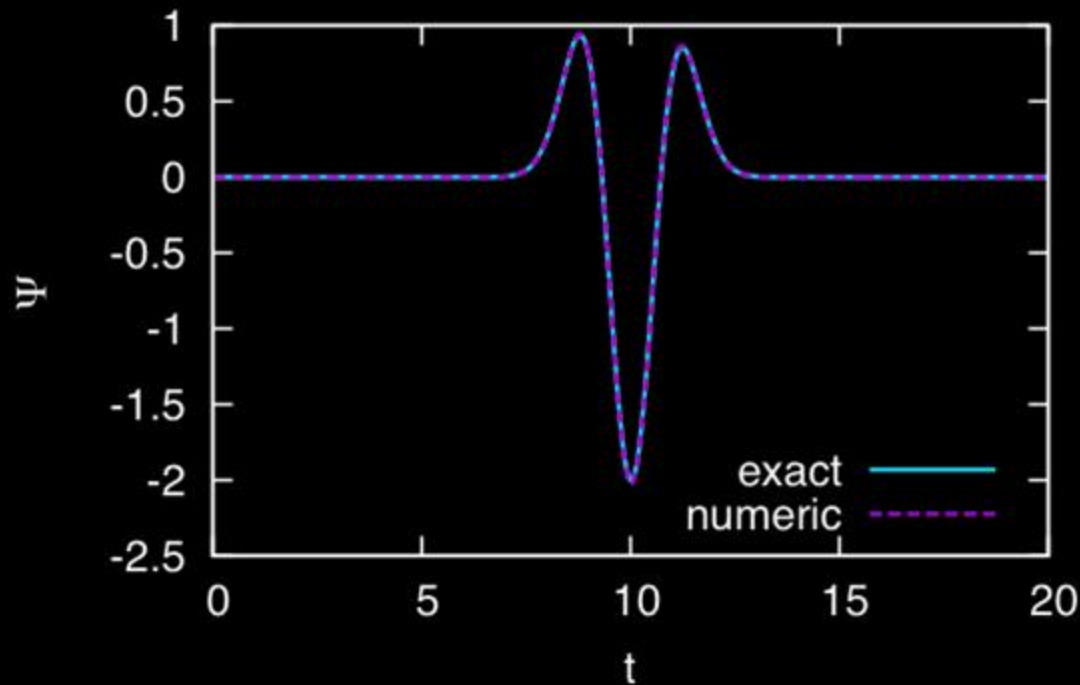
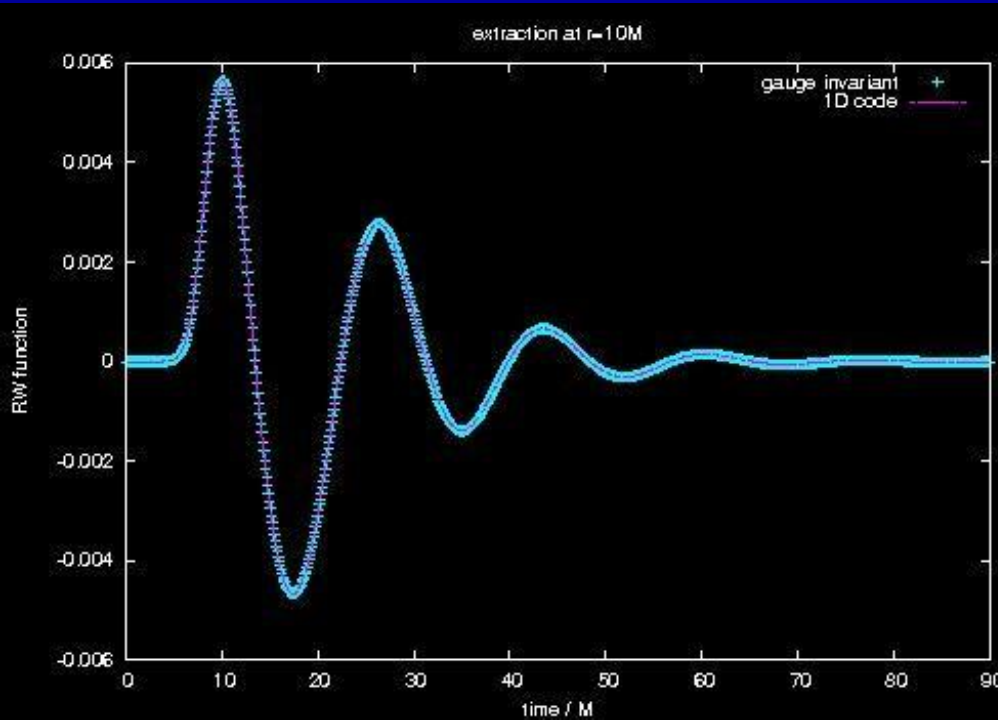# 3D weak gravitational waves

Odd parity perturbations of flat spacetime (sort of Teukolsky waves). The Regge-Wheeler equation in this case can be exactly solved and the numerical solution can be compared to the exact (linear) one.

Scaled errors in the wave form

3D distorted black holes

- Odd parity distortions of a 3D Schwarzschild black hole.
- At the linear level, the problem can be solved using a generalized gauge invariant perturbation formalism (resulting in 1+1 equations which can be solved with very high resolution)

3D distorted black holes