Viscosity driven instability in rotating relativistic stars

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CONTENTS

1. Introduction

Secular and dynamical bar mode instability

Magnitude of Rotation

$$\beta \equiv T/W$$

T: Rotational kinetic energy
W: Gravitational binding energy

Toroidal mode analysis in Maclaurin spheroid (Chandrasekhar 69)

Lagrangian displacement

$$\xi(t, \boldsymbol{x}) = e^{i\omega t}\xi(\boldsymbol{x})$$

$\omega = 0$

Secular instability

$$\beta_{\mathbf{sec}} \simeq 0.14$$

$$\tau_{\mathbf{sec}} \sim \tau_{\mathbf{vis}} \text{ or } \tau_{\mathbf{gw}}$$

Neutral point
Star becomes unstable when we take dissipation into account

$\omega^2 = 0$

Dynamical instability

$$\beta_{\mathbf{dyn}} \simeq 0.27$$

$$\tau_{\mathbf{dyn}} \sim (G\bar{\rho})^{-1/2} \ll \tau_{\mathbf{sec}}$$

Dynamically unstable
Star becomes unstable due to hydrodynamics (rotation)

β

Secular instability in the low temperature limit

- Viscosity driven instability (Roberts,Stewartson 1963)

  Configuration transits to lower energy state due to viscosity
  Sets in when a mode has a zero-frequency in the frame rotating with the star

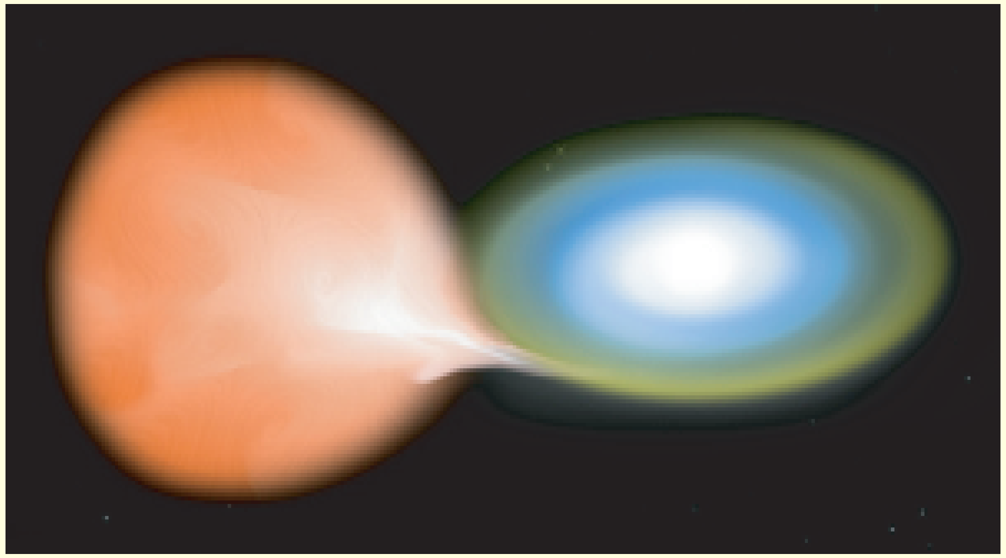
  $$\omega + m\Omega = 0$$ m=-2 mode

Astrophysical scenario

$$t_{\rm ev} > t_{\rm vis}, t_{\rm mag}$$

High viscosity
Strong magnetic field ➡ Maintains uniform rotation

Accreting neutron star

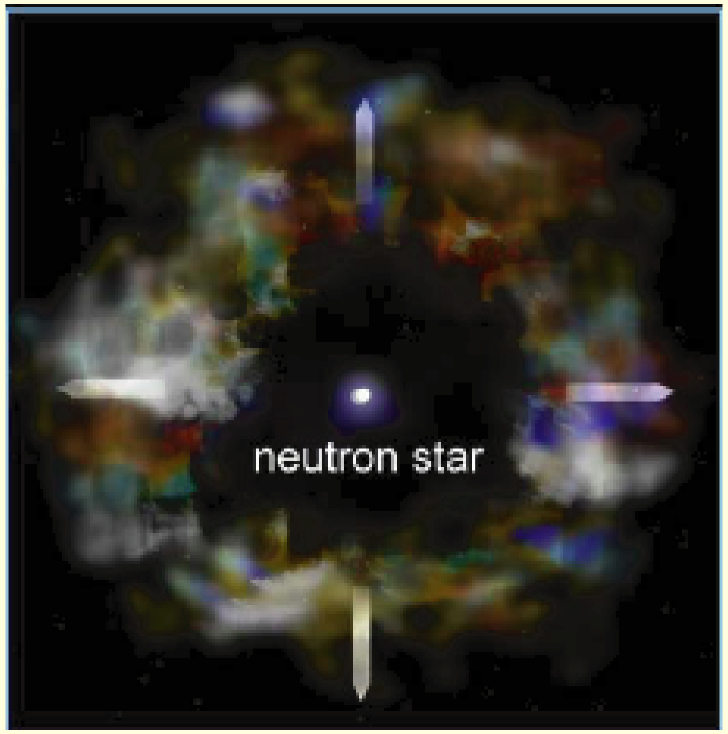


@NGSL, Michigan

- Gravitational wave driven instability (Chandrasekhar 70, Friedman Schutz 78)

Configuration transits to lower energy state due to gravitational radiation
Sets in when the backward going mode is dragged forward in the inertial frame

$$\omega - m\Omega = 0$$ m=2 mode



@Chandra web site

Astrophysical scenario

$$t_{\rm ev} < t_{\rm vis}, t_{\rm mag}$$

Low viscosity
Magnetic braking ➡ Leads to differential rotation

Newly born neutron star

The opponent effects competes with each other !

(Detweiler, Lindblom 1977)

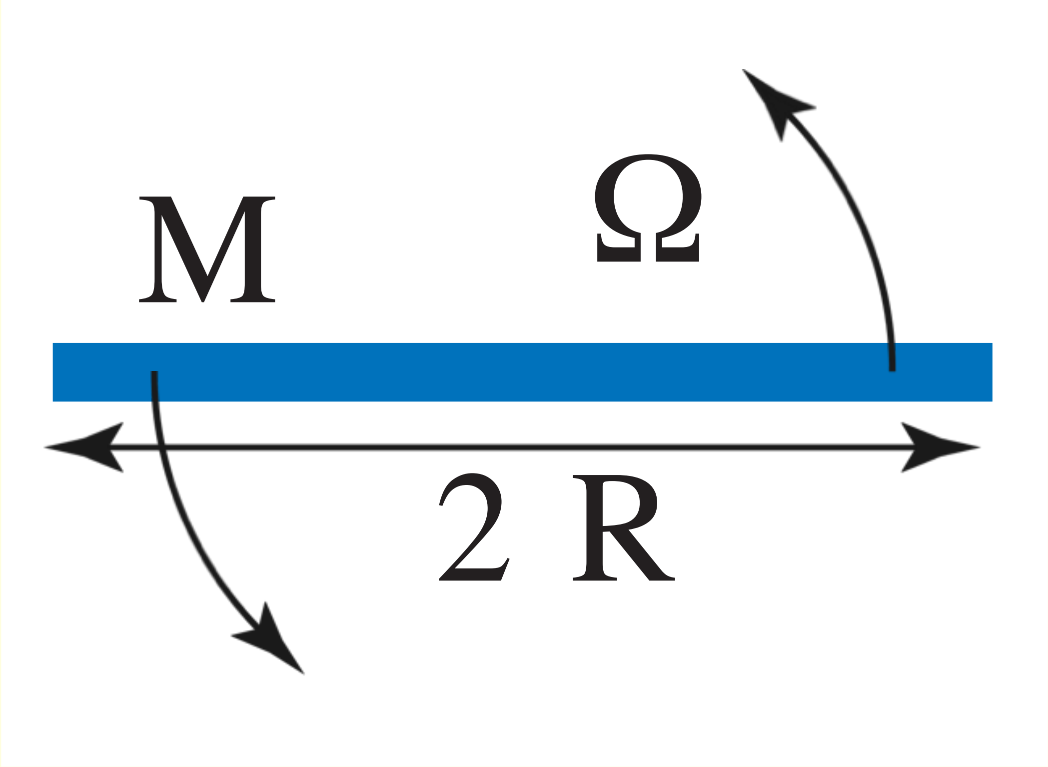
e.g. Viscosity driven instability is stablized by the gravitational radiation

Gravitational waves from bar mode

Mechanism

Global rotational instabilities in fluids arise from nonaxisymmetric bar mode

Quadrupole formula



Energy Flux

$$\frac{dE}{dt} = \frac{8}{45} M^2 R^4 \Omega^6 \sim \left(\frac{M}{R}\right)^5 \left(\frac{T}{W}\right)^3$$

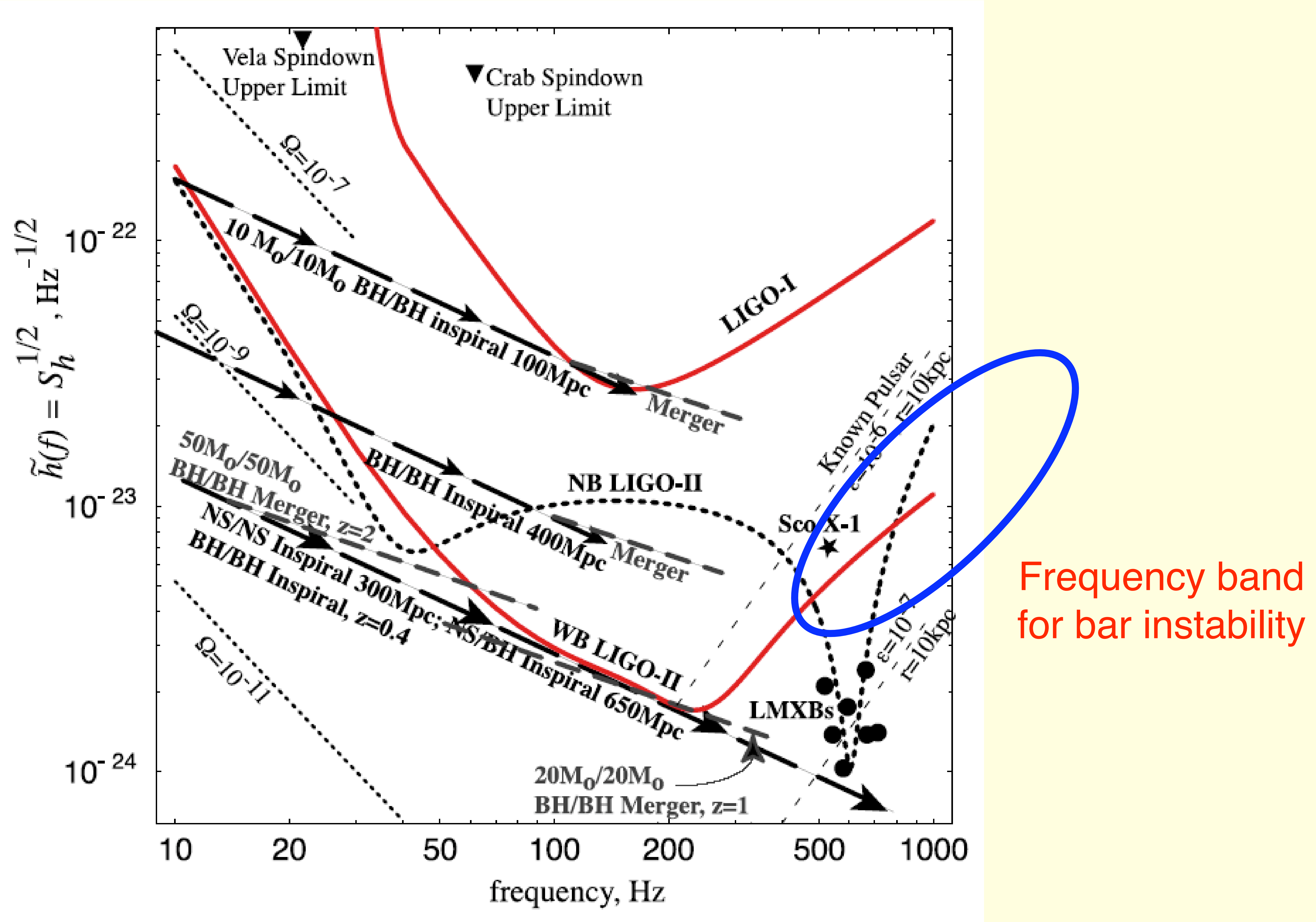
Gravitational Waveform

$$rh_+ = \frac{1}{2}\frac{d^2}{dt^2}(I_{xx} - I_{yy}) = -\frac{2}{3} M R^2 \Omega^2 \cos 2\Omega t$$

$$rh_\times = \frac{d^2}{dt^2} I_{xy} = -\frac{2}{3} M R^2 \Omega^2 \sin 2\Omega t$$

Feature

Frequency is Ω/π since the bar spins its center of mass



(Cutler & Thorne 2002)

2. Bifurcation theory in Newtonian gravity

Riemann S-type ellipsoid

Nonaxisymmetric body with one principal rotational axis (including a uniform vorticity)

$$f \equiv \frac{\zeta}{\Omega} = \mathrm{Const.} \qquad\qquad x = \frac{ab}{a^2 + b^2} f$$

- Maclaurin spheroid

  Uniformly rotating axisymmetric
- incompressible body

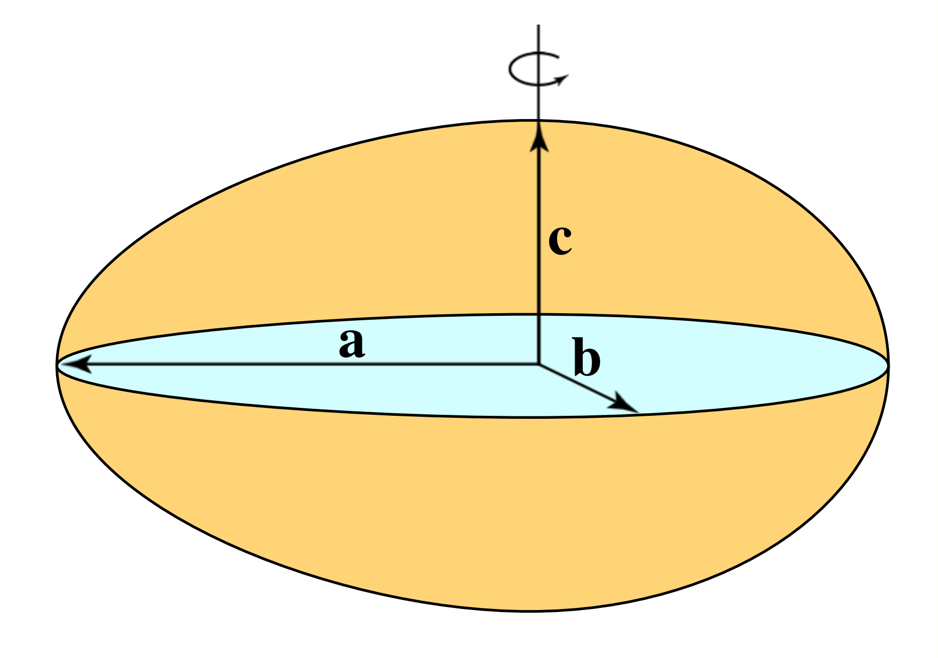
  $$a = b, \Omega = \Omega_c \qquad\qquad f = 0$$
- Jacobi ellipsoid

  Uniformly rotating nonaxisymmetric
- incompressible body

  $$\Omega = \Omega_c \qquad\qquad f = 0$$
- Dedekind ellipsoid

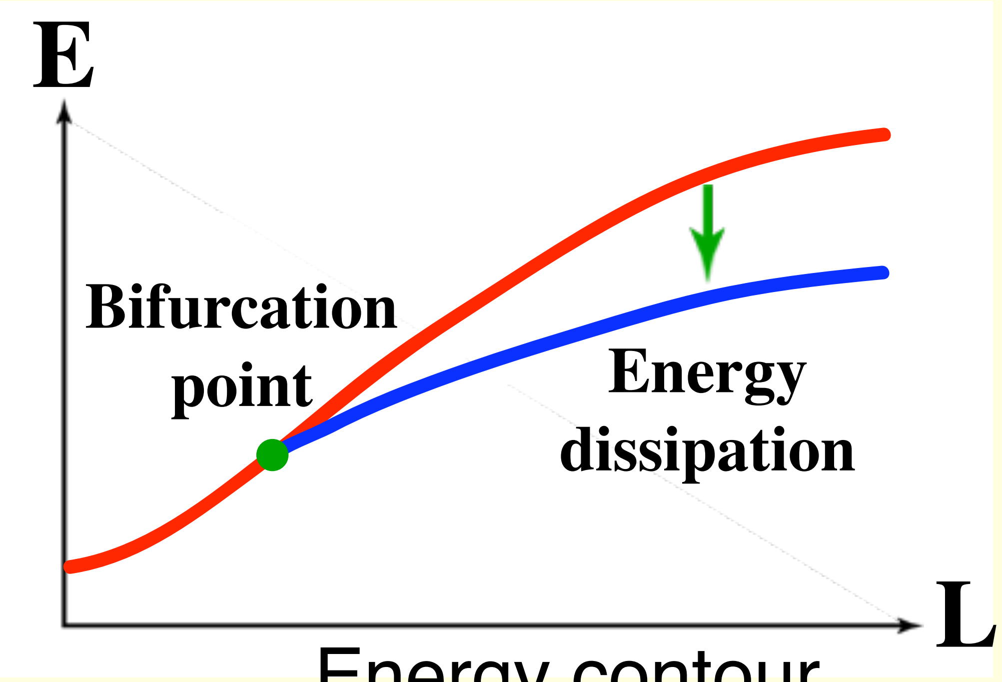
  Differentially rotating nonaxisymmetric incompressible body

  $$f = \pm\infty$$



Meaning of the secular instability

(Christodoulou et al. 1995)



- Solely L conservation

  Free-energy function E:

$$E = \frac{L^2}{2}\frac{(a+bx)^2+(b+ax)^2}{a^2+b^2+2abx} - 2I(a,b,c)$$

$$\frac{\partial E}{\partial x} = \frac{L^2(a-b)^2 x}{(a^2+b^2+2abx)^3}$$

L: Angular momentum

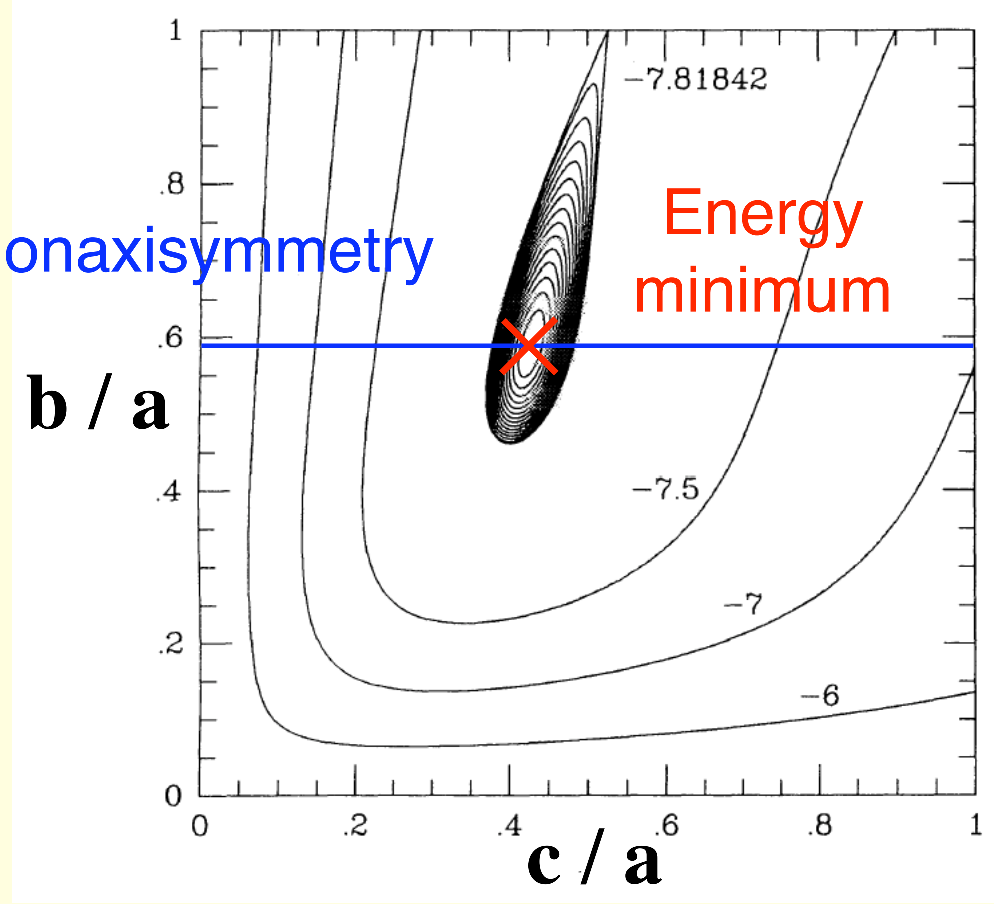
I: Moment of inertia

Energy minimum at $x = 0$

(2nd order derivative in x is possitive)

➡ $\boxed{\text{Jacobi ellipsoid}}$

Energy contour



Features

1. Maclaurin spheroid is the energy minimum state up to the bifurcation point
2. Jacobi ellipsoid is the energy minimum state beyond the bifurcation point through the variation of circulation
3. Bifurcation point corresponds to the neutral point

Violation of the circulation is induced by viscous dissipation

- Solely C conservation

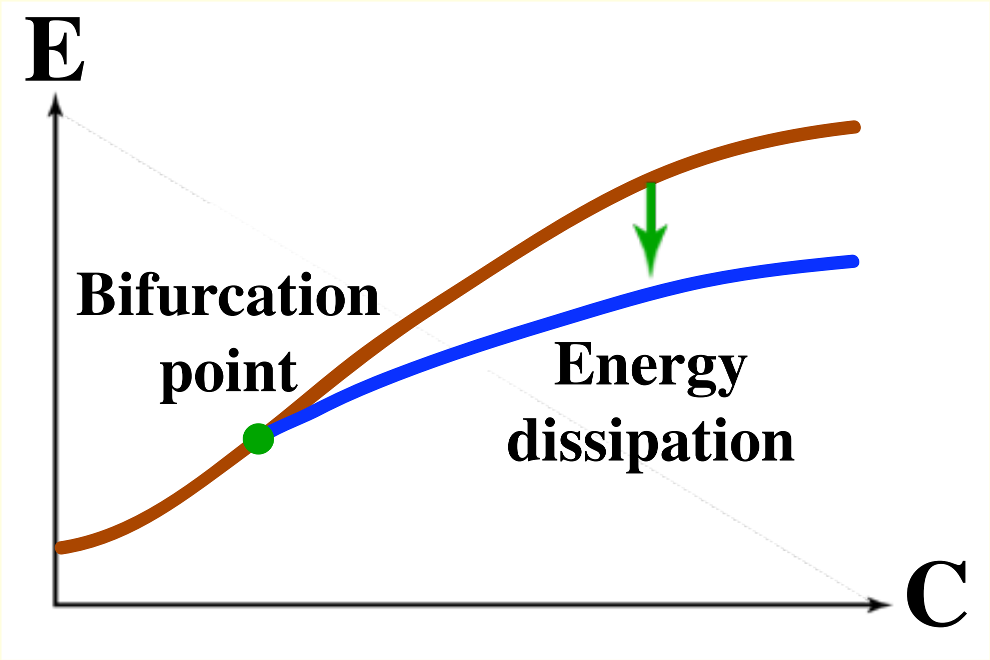
Free-energy function E:

$$E = \frac{C^2}{2}\frac{(a+bx)^2+(b+ax)^2}{[2ab+(a^2+b^2)x]^2} - 2I(a,b,c) \qquad \frac{\partial E}{\partial x'} = \frac{C^2(a^2-b^2)^2 x'}{(a^2+b^2+2abx')^3}$$ where $x' = 1/x$

Energy minimum at
(2nd order derivative in x' is possitive) ➡ Dedekind ellipsoid

Features

1. Maclaurin spheroid is the energy minimum state up to the bifurcation point
2. Dedekind ellipsoid is the energy minimum state beyond the bifurcation point through the variation of angular momentum
3. Bifurcation point corresponds to the neutral point



Violation of the angular momentum is induced by gravitational radiation

3. Bifurcation theory in general relativity

Nonaxisymmetric spacetime

$$ds^2 = -N^2 dt^2 + A^2(dr - N^r dt)^2 + r^2 A^2 (d\theta - N^\theta)$$
$$+ r^2 \sin^2\theta B^2 (d\varphi - N^\varphi dt)^2$$

N : lapse

N^i : correspond to shift

A, B : function of the spatial metric

Only true when the azimuthal variable is separable

Equilibrium state

$N^r, N^\theta = 0$ and all the functions only depend on (r, θ)

➡ Axisymmetric spacetime in quasi-isotropic coordinate

Nonaxisymmetric perturbation

$$\ln N = \ln N_{\mathrm{eq}}(1 + \epsilon \sin^2\theta \cos 2\varphi)$$

Treatment of the spacetime

1. Lapse and shift depend on (r, θ, φ)
2. A and B depend on (r, θ)

➡ Fully constraint equations to be solved
Consistent up to 1/2 PN order

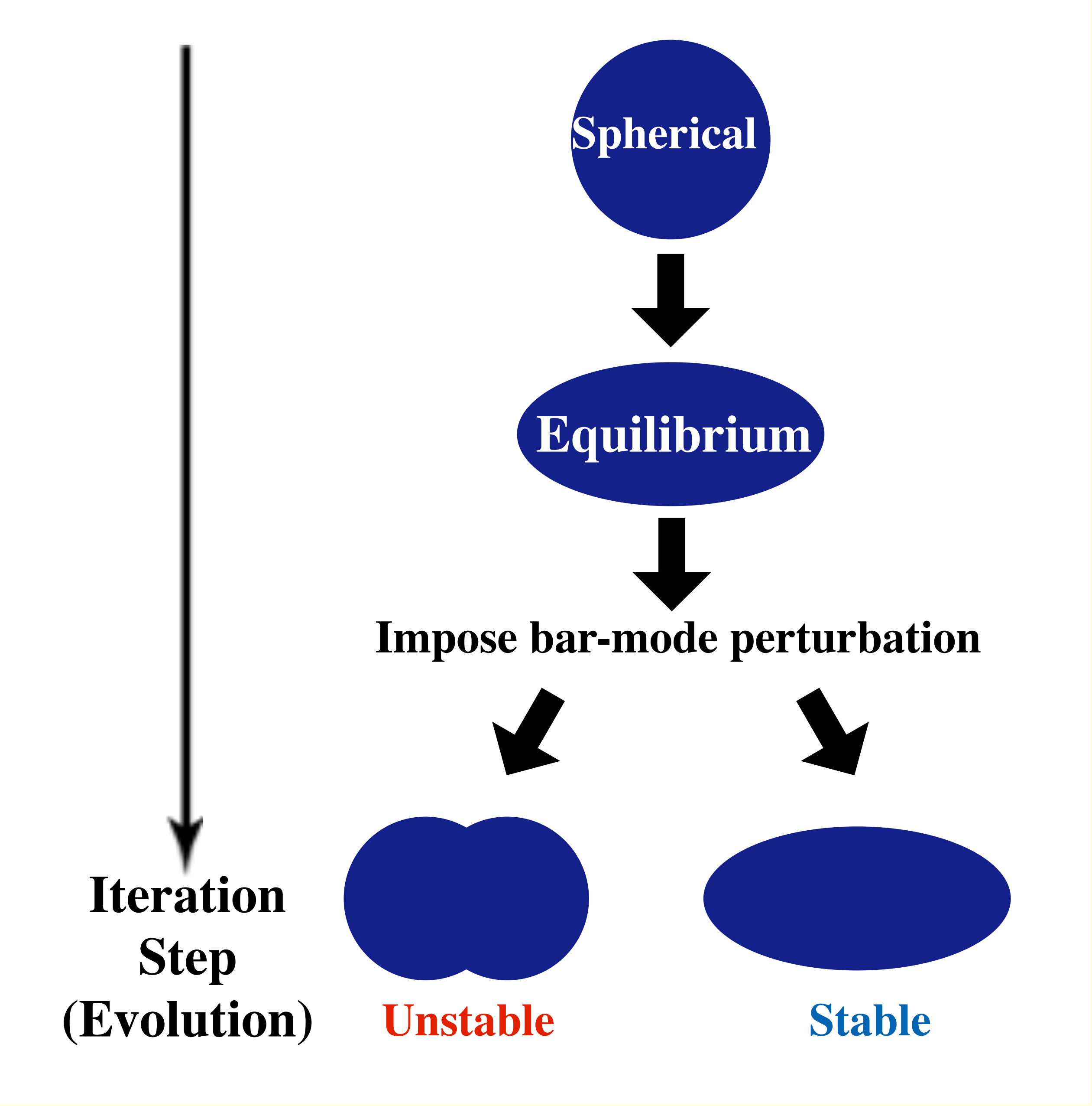
$$\epsilon \times \left(\frac{M}{R}\right) \sim 10^{-6}$$

satisfactory approximation!

Iterative evolution approach

(Bonazzolla, Frieben, Gourgoulhon 96, 98)

Investigate nonaxisymmetric instability in quasi-static evolution in general relativity



Advantage

- No restriction to the iteration (time) step
- Fully constraint scheme
- Coincidence of a bifurcation point in Newtonian incompressible star
  (Gondek-Rosinska, Gourgoulhon 03)

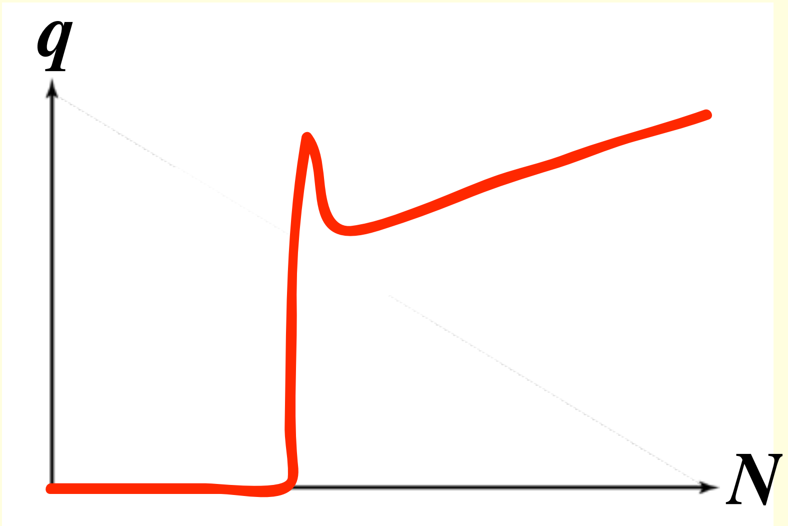
Disadvantage

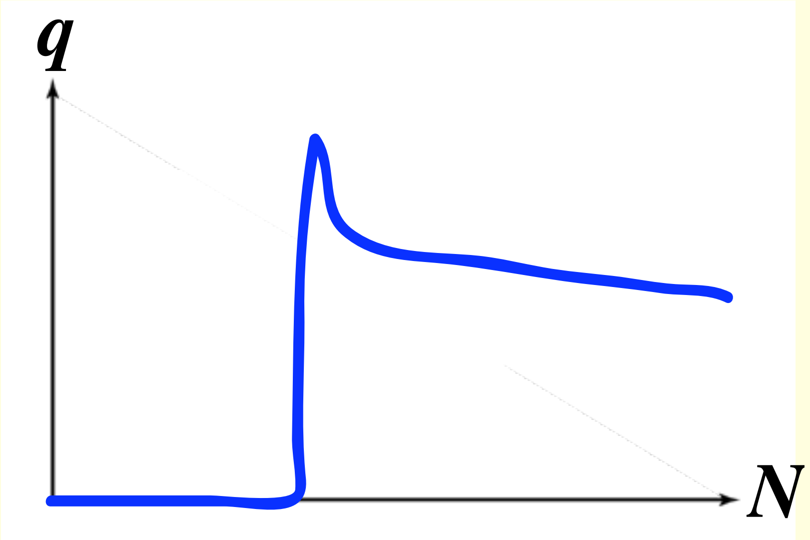
- The direction of time evolution is not clear in a strict sense
- Restriction to the axisymmetric fluid flow

Diagnostics

$$q = \max|\ln \hat{N}_2| \quad \text{where} \quad \ln N_2 = \sum_{m=0}^{\infty} \ln \hat{N}_m e^{im\varphi}$$

Illustration

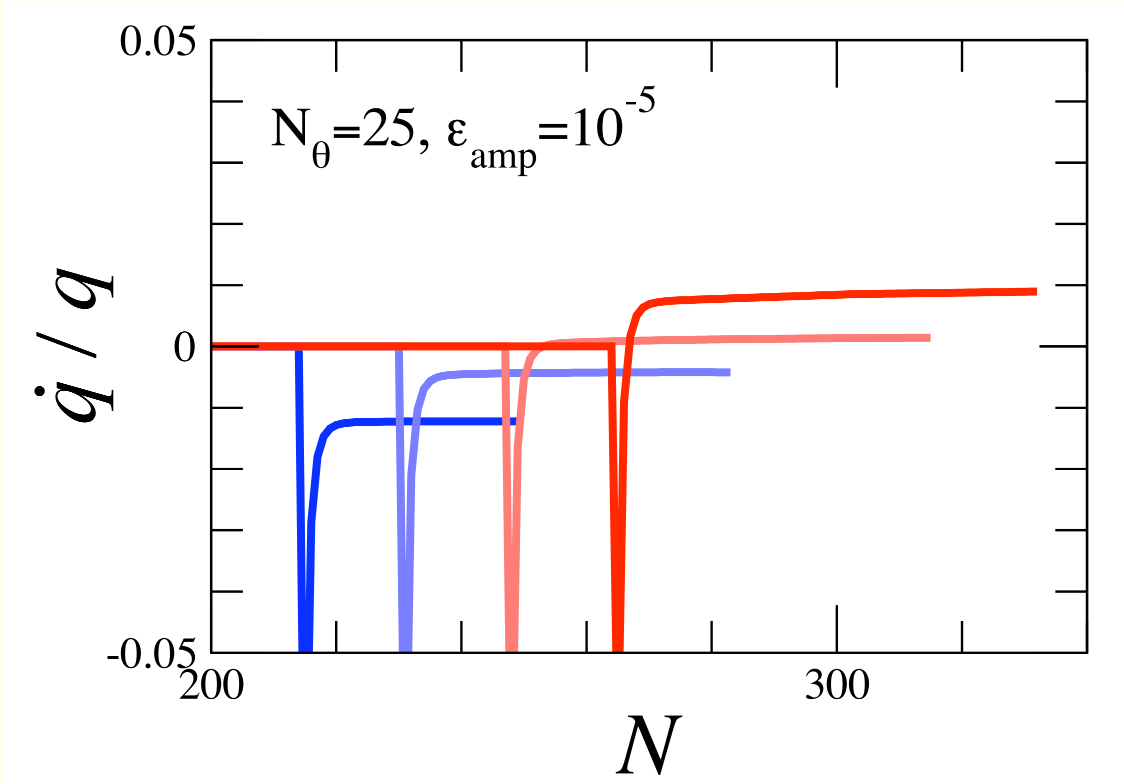




Unstable: q grows exponentially through iteration

Stable: q decays exponentially through iteration

Precise measurement



Investigate the logarithmic derivative of q

Unstable: Positive value
Stable: Negative value

4. Viscosity driven instability in rotating relativistic stars

Incompressible stars

(Gondek Rosinska, Gourgoulhon 2002)

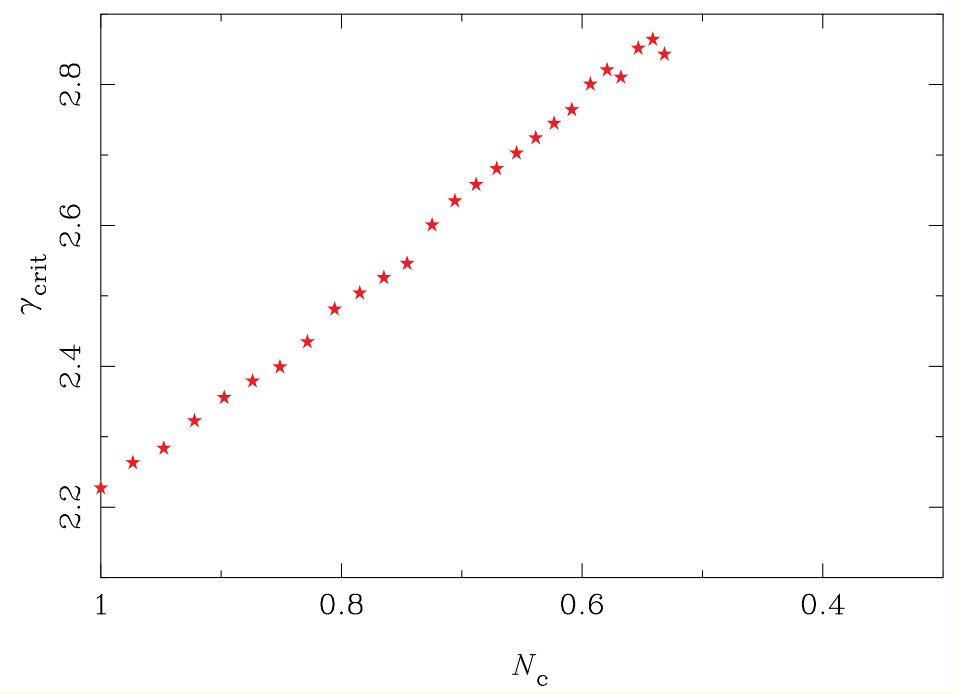
M/R	---	0.0227	0.0438	0.0984	0.1556	0.2000	0.2430
T/W	0.1375	0.1412	0.1446	0.1539	0.1642	0.1729	0.1822

Relativistic gravitation stabilizes the system from viscosity driven instability

The above statement also agrees with the pN results in incompressible stars (Di-Girolamo, Vietri 02)

Rigidly rotating polytropic stars

(Bonazzola, Frieben, Gourgoulhon 98)



Investigate the stability at mass-shedding limit

Relativistic gravitation stabilizes the system from viscosity driven instability

Rigidly rotating stars in Newtonian gravity

(Bonazzola, Frieben, Gourgoulhon 1996)

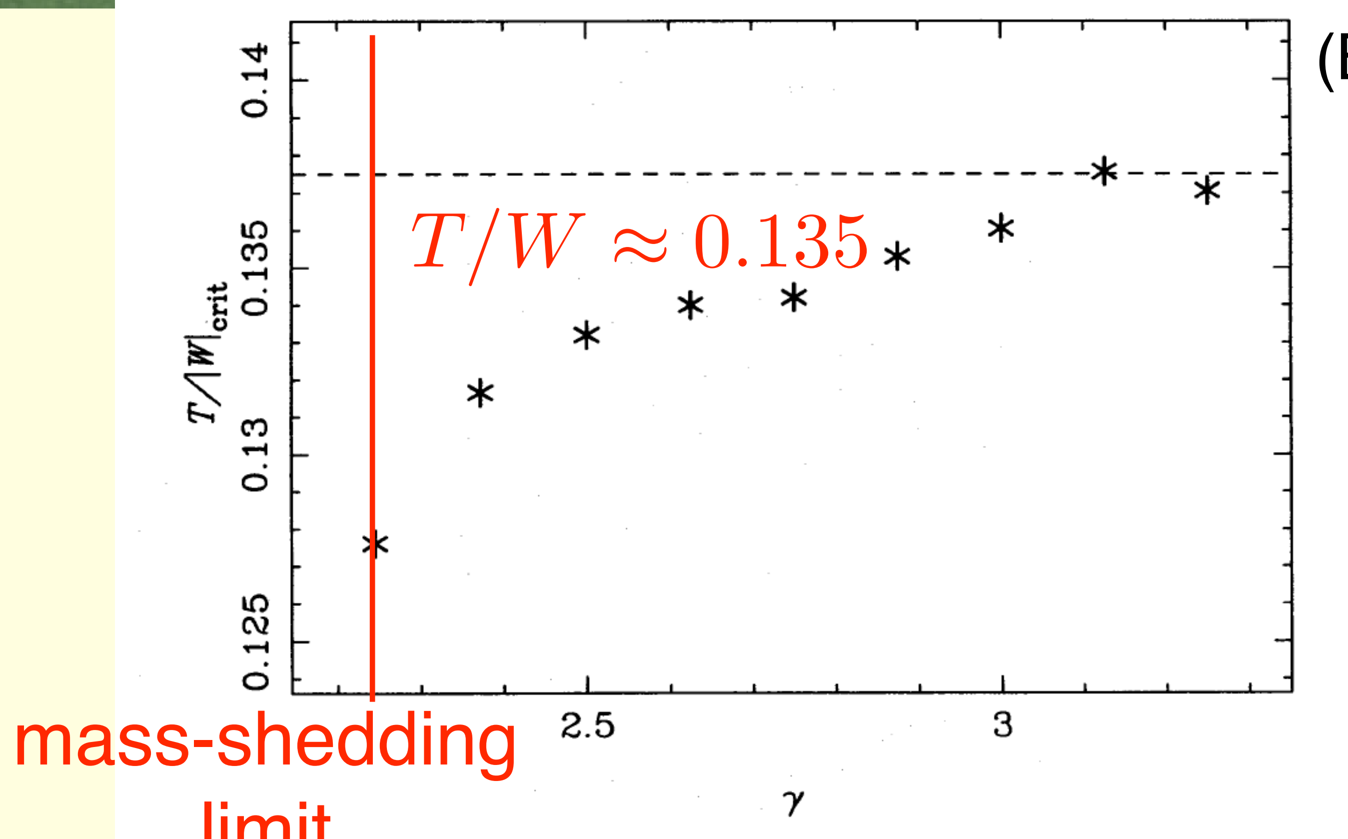
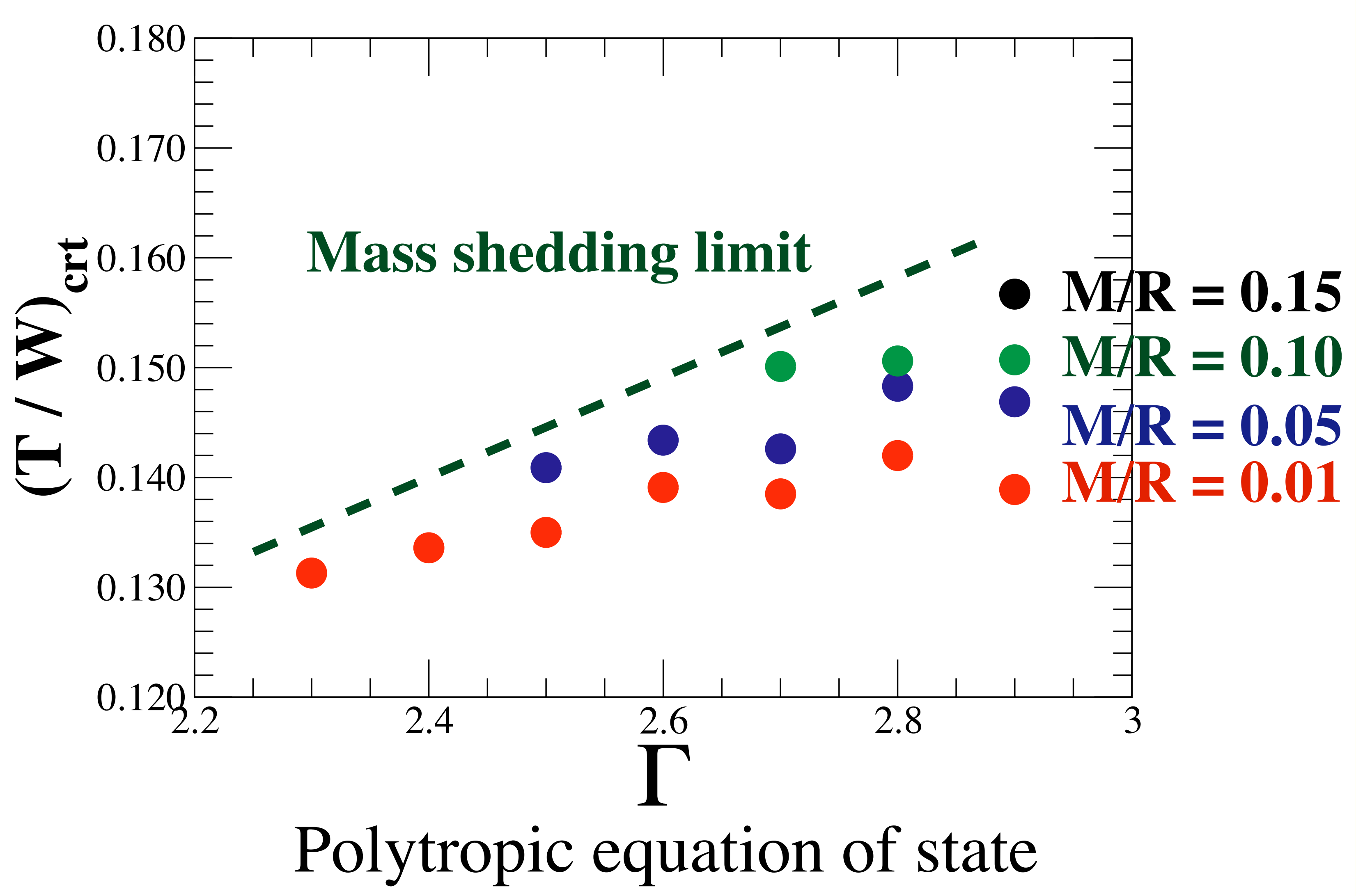


FIG. 3.—Ratio of the kinetic energy T to the gravitational potential energy W at the triaxial Jacobi-like bifurcation point along a sequence of rotating Newtonian polytropes, as a function of the adiabatic index γ. The dashed horizontal line corresponds to the theoretical value of $T/|W|$ for incompressible Maclaurin spheroids.

Compressible stars have slightly lower criterion of T/W than in the incompressible star

Rigidly rotating stars in general relativity

(MS, Gourgoulhon 06)



- Relativistic gravitation stabilizes viscosity driven instability
- The bifurcation point is not so sensitive to the stiffness of the equation of state

Differentially rotating stars in general relativity

- Due to viscous friction, the angular momentum distribution should be changed
- We assume that it is small and still remains the present angular momentum distribution

1. Fixed rotation profile

$$\Gamma = 2$$

Rotation raw (equilibrium state)

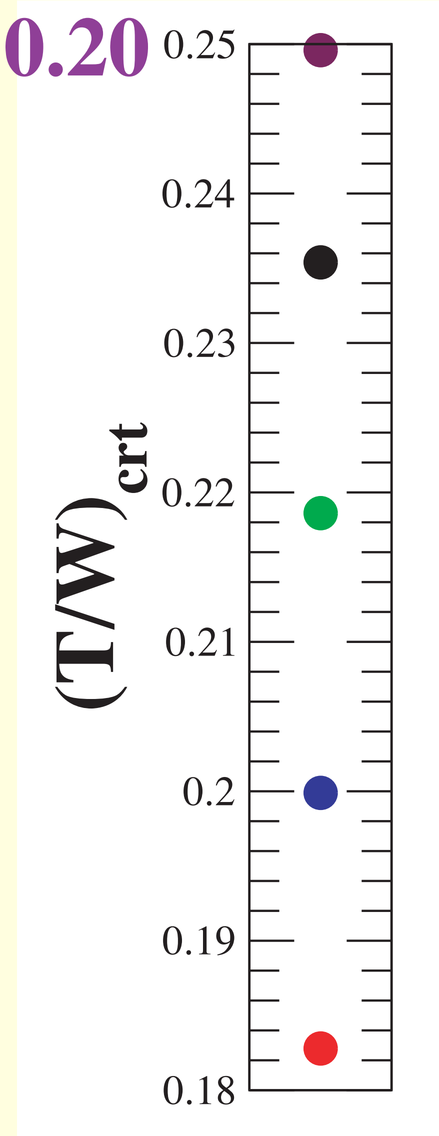
$$\Omega \approx \frac{A^2 \Omega_0}{\varpi^2 + A^2} \qquad A = R_{\rm e} \longrightarrow \Omega_0/\Omega_{\rm eq} \sim 2$$

Relaxes the restriction of the mass-shedding limit in rigid rotation

Differentially rotation also stabilizes the sytem

But do you believe this current result?

M/R = 0.20



2. Varied rotation profile

Helical Killing vector

$$k^\mu = \xi^\mu + \Omega \chi^\mu$$

ξ^μ : timelike Killing vector
χ^μ : rotational Killing vector

Ω = Constant $\Rightarrow$ stationary
otherwise static configuration

But lazy physicist ...

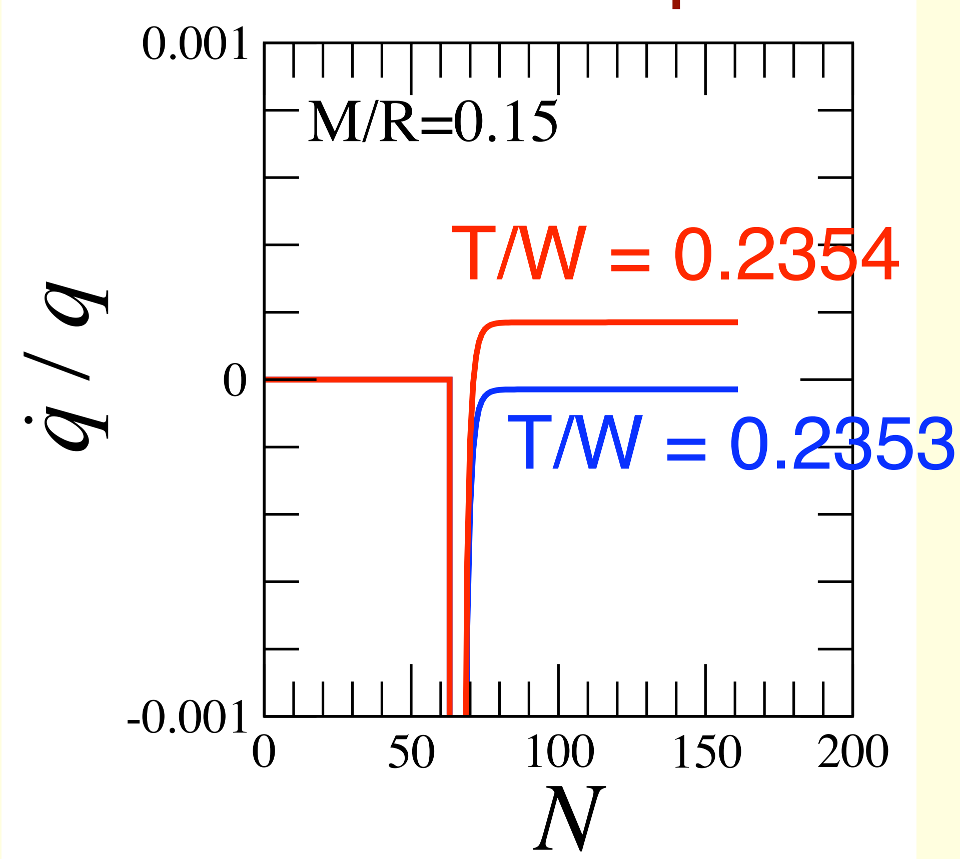
Variation of rotation profile

$$A_{\rm rot}^{-1} = A_{\rm rot}^{-1(\rm eq)}[1 - \varepsilon_{\rm rot}(\mathcal{N} - \mathcal{N}_{\rm ptb})]$$

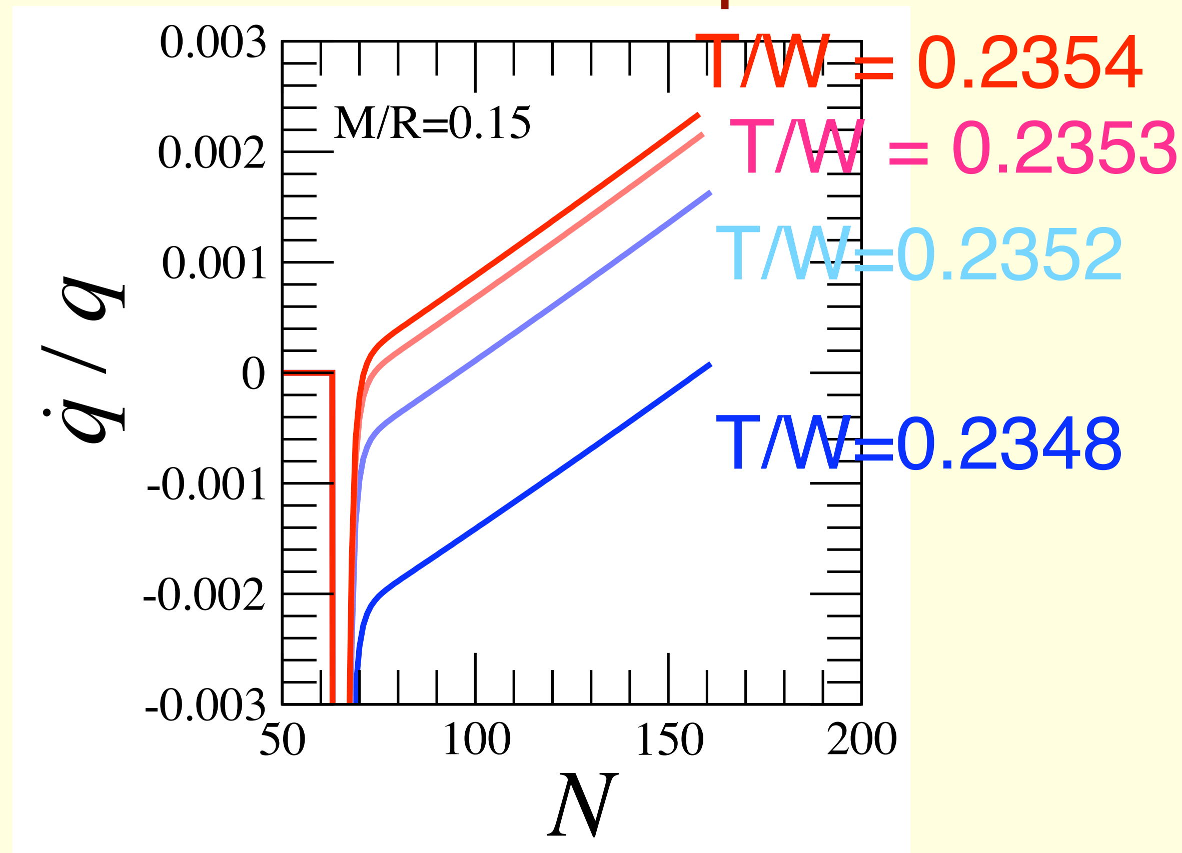
$$\Omega_{\rm c} = \Omega_{\rm c}^{(\rm eq)}[1 - \varepsilon_{\rm omg}(\mathcal{N} - \mathcal{N}_{\rm ptb})]$$

Adjust $\varepsilon_{\rm omg}/\varepsilon_{\rm rot}$ to maintain J conservation throughout the iteration

Fixed rotation profile



Varied rotation profile



All T/W around the threshold in fixed rotation profile become unstable due to the change of angular momentum distribution

Timescales from the computational results

$$\tau_{\rm bar} = (\dot{q}/q)^{-1}$$

$$\tau_{\rm ang} = (\ddot{q}/q)^{-1/2}$$

T/W	$\tau_{\rm ang}$	$\tau_{\rm bar}$
0.2348	2.0E2	-4.8E2
0.2352	2.0E2	-1.7E3
0.2353	1.9E2	-3.4E4
0.2354	1.9E2	5.9E3

Timescales based on Newtonian Navie-Stokes equation

- E-folding time of the variation of rotation profile
- $$\tau_{\rm ang} = \frac{R^2}{8\nu} \frac{\Omega_{\rm c}}{\Omega_{\rm c} - \Omega_{\rm s}}$$
- Growth timescale of the bar mode

$$\tau_{\rm bar} = \frac{\kappa_n R^2}{5\nu} \frac{\beta_{\rm sec}}{\beta - \beta_{\rm sec}}$$

Adjusted timescales for computation

$$\tau_{\rm bar} \approx \epsilon_{\rm org}^{-1} \left(\frac{\Omega_{\rm c} - \Omega_{\rm s}}{\Omega_{\rm c}} \right) \left(\frac{\beta_{\rm sec}}{\beta - \beta_{\rm sec}} \right)$$

$$\tau_{\rm ang} \approx \epsilon_{\rm org}^{-1}$$

Taking into account of the table, the deviational ratio of T/W from the one of fixed rotational profile is roughly the same order of $\approx \varepsilon_{\rm org}^{-1}$

5. Summary

We study viscosity driven instability in both uniform and differential rotating polytropic stars by means of iterative evolution approach in general relativity

- Relativistic gravitation stabilizes from the viscosity driven instability, with respect to the Newtonian gravity

- Differential rotation also stabilizes the star significantly from the viscosity driven instability, even we take the effect of angular momentum distribution into account

- Gravitational waves can be detected in Advanced LIGO, but require some spin-up process of neutron stars in low temperature regime