Outer Boundary Conditions for the Generalized Harmonic Einstein Equations: Stability and Accuracy

Oliver Rinne

Work with the Caltech-Cornell Numerical Relativity Collaboration

Theoretical Astrophysics and Relativity, California Institute of Technology

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Outline

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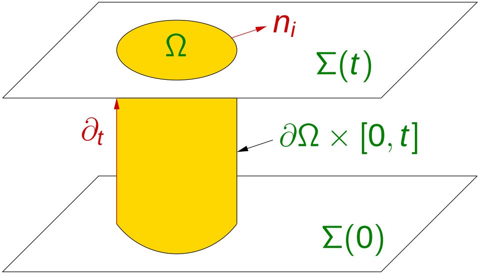
The initial-boundary value problem

- Consider Einstein's equations on compact spatial domain Ω with smooth outer boundary $\partial\Omega$



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  1. yield a well-posed initial-boundary value problem
  2. be compatible with the constraints (*constraint-preserving*)
  3. minimize reflections, control incoming gravitational radiation

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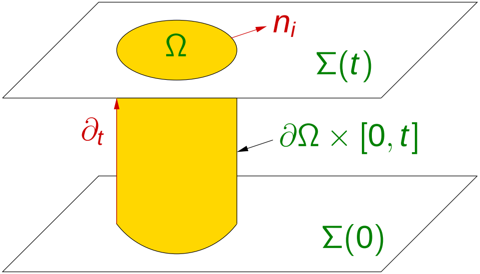

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- [Friedrich & Nagy 1999] formulation that satisfies all three requirements for the fully nonlinear vacuum Einstein equations (tetrad-based, evolves Weyl tensor)
- Necessary conditions for well-posedness can be verified using pseudo-differential techniques (Fourier-Laplace analysis) [Stewart 1998, Calabrese & Sarbach 2003, Sarbach & Tiglio 2005, Kreiss & Winicour 2006, R 2006]
- Alternate approach to proving well-posedness via semigroup theory [Reula & Sarbach 2005, Nagy & Sarbach 2006]
- Improved absorbing boundary conditions [Lau 2004-5, Novak & Bonazzola 2004, Buchman & Sarbach 2006]
- Some alternatives: spatial compactification, Cauchy-characteristic and Cauchy-perturbative matching, hyperboloidal slices, . . .

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(Generalized) harmonic gauge

- Harmonic coordinates

$$\Box x^a = 0$$

- Principal part of Einstein equations becomes wave operator on metric ψ_{ab},

$$0 = R_{ab} \simeq -\tfrac{1}{2}\Box\psi_{ab}$$

- Symmetric hyperbolic system, Cauchy problem is well-posed [Choquet-Bruhat 1952]
- Subject to constraints

$$\mathcal{C}_a \equiv H_a - \Box x_a = H_a + \Gamma_{ab}{}^{b} = 0$$

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First-order reduction

- [Lindblom *et al.* 2006] Introduce new variables for first time and spatial derivatives of metric

$$\Pi_{ab} \equiv -t^c \partial_c \psi_{ab}, \qquad \Phi_{iab} \equiv \partial_i \psi_{ab}$$

(t^a normal to $t = \text{const.}$ hypersurfaces, indices $i, j, \ldots = 1, 2, 3$)

- New constraints

$$\mathcal{C}_{iab} \equiv \partial_i \psi_{ab} - \Phi_{iab} = 0, \quad \mathcal{C}_{ijab} \equiv 2\partial_{[i}\Phi_{j]ab} = 0$$

- To principal parts, obtain

$$\begin{aligned}
\partial_t \psi_{ab} &\simeq 0, \\
\partial_t \Pi_{ab} &\simeq N^k \partial_k \Pi_{ab} - N g^{ki} \partial_k \Phi_{iab} + \gamma_2 N^k \partial_k \psi_{ab}, \\
\partial_t \Phi_{iab} &\simeq N^k \partial_k \Phi_{iab} - N \partial_i \Pi_{ab} + N \gamma_2 \partial_i \psi_{ab},
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Characteristic structure

- System is symmetric hyperbolic, characteristic variables in direction n_i are

$$\begin{aligned} u^0_{ab} &= \psi_{ab}, & &\text{speed } 0, \\ u^{1\pm}_{ab} &= \Pi_{ab} \pm \Phi_{nab} - \gamma_2 \psi_{ab}, & &\text{speed } -N^n \pm N, \\ u^2_{Aab} &= \Phi_{Aab}, & &\text{speed } -N^n \end{aligned}$$

($v_n \equiv n_i v^i$, $v_A \equiv P_{Ai} v^i$, boundary metric $P_{ij} \equiv g_{ij} - n_i n_j$)

- Note dependence of speeds on normal component N^n of shift

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Constraint-preserving boundary conditions

- Constraints obey subsidiary system

$$\Box \mathcal{C}_a \simeq 0$$

- Set incoming modes of this system to zero at the boundary (in contrast, [Kreiss & Winicour 2006] use $\mathcal{C}_a \doteq 0$)
- Obtain conditions on *normal derivatives* of 4 components of main incoming fields u^{1-},

$$P^{\mathrm{C}\,cd}_{ab} \partial_n u^{1-}_{cd} \doteq \text{(tangential derivatives)},$$

  where P^{C} is projection operator with rank 4
- If $N^n \dot{>} 0$ then u^2_{Aab} also need boundary conditions, obtained by requiring

$$\mathcal{C}_{nAab} \doteq 0 \Rightarrow \partial_n \Phi_{Abc} \doteq \partial_A \Phi_{nbc}$$

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Physical boundary conditions

- Incoming gravitational radiation ⇔ Newman-Penrose scalar

$$\Psi_0 = C_{abcd} l^a m^b l^c m^d,$$

$\{l^a = (t^a + n^a)/\sqrt{2}, k^a = (t^a - n^a)/\sqrt{2}, m^a, \bar{m}^a\}$ complex null tetrad

- We impose the BC

$$\Psi_0 \doteq 0$$

- Rewrite as

$$P^{\mathrm{P}}{}_{ab}{}^{cd} \partial_n u^{1-}_{cd} \doteq (\text{tangential derivatives}) + h^{\mathrm{P}}_{ab},$$

where P^{P} has rank 2 and is orthogonal to P^{C}

- Lowest level in a hierarchy of perfectly absorbing BCs for linearized gravitational waves [Luisa Buchman's talk]

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Gauge boundary conditions

- Remaining gauge freedom $x^a \to x^a + \xi^a$ provided that

$$\Box \xi^a = 0$$

- Induced metric change

$$\psi_{ab} \to \psi_{ab} - 2\partial_{(a}\xi_{b)}$$

- Ideally, impose absorbing BC on ξ^a
- To leading order in inverse radius, a suitable BC is

$$P^{\mathrm{G}\,cd}_{ab}(u^{1-}_{cd} + \gamma_2 \psi_{ab}) \doteq 0$$

where P^{G} has rank 4 and $P^{\mathrm{C}} + P^{\mathrm{P}} + P^{\mathrm{G}} = \mathbb{I}$

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2. Construction of boundary conditions
3. Stability analysis
4. Accuracy comparisons
5. Summary

Fourier-Laplace analysis

- Consider high-frequency perturbations about any given spacetime
- Obtain linear symmetric hyperbolic system with constant coefficients
- Solve by Laplace transform in time and Fourier transform in space
- Boundary conditions imply linear system of equations for integration constants
- Study zeros of its (complex) determinant $\Rightarrow$ necessary conditions for well-posedness (determinant condition and Kreiss condition)
- GH system satisfies both conditions [R 2006]

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Towards sufficient conditions

- Kreiss condition implies that solution can be estimated in terms of boundary data (boundary-stable)
- One would also like to control
  - source terms (well-posedness in the generalized sense)
  - initial data (well-posedness)
- Proof via symmetrizer construction [Kreiss 1970]
- Technique not applicable to boundary conditions of differential type
- Can show that system is free of weak instabilities with polynomial time dependence

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Numerical robust stability test

- Consider fixed background solution (Minkowski or Schwarzschild)
- Add small random perturbations to *initial data, boundary data and right-hand-sides of evolution equations*
- Evolve on domain $T^2 \times \mathbb{R}$, impose BCs in transverse direction
- Pseudospectral collocation method [Caltech-Cornell Spectral Einstein Code]
- Monitor error (deviation from background solution) and constraint violations

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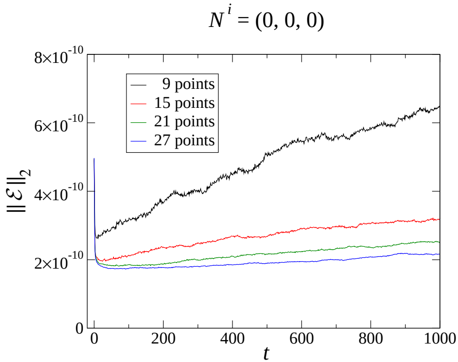
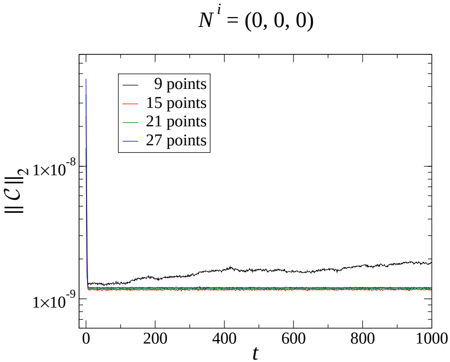
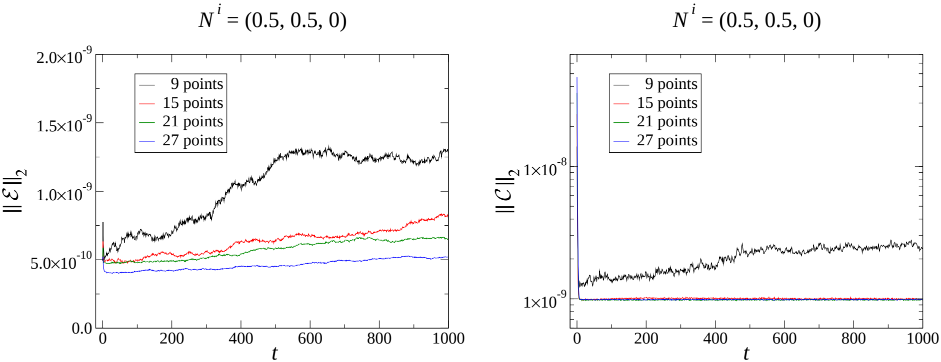
Flat space without shift





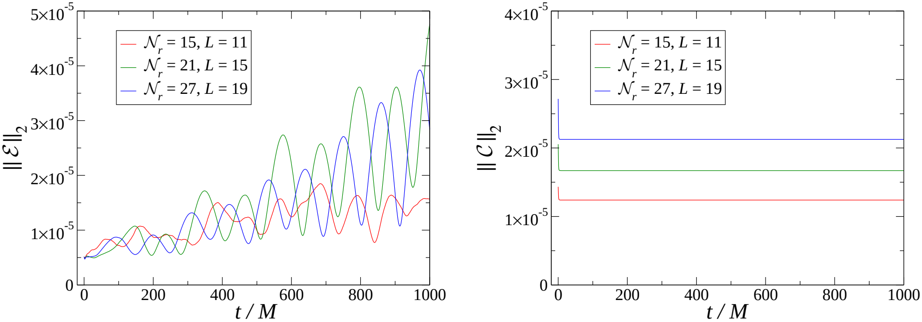
Random data amplitude 10^{-10}

Flat space with constant shift



Random data amplitude 10^{-10}

Schwarzschild



Random data amplitude 10^{-6}

Outline

Some alternative boundary treatments

- Freezing all the incoming fields

$$u^{1-}_{ab} \doteq 0 \qquad (\text{and} \quad u^{2}_{Aab} \doteq 0 \quad \text{if} \quad N^n \dot{>} 0)$$

- Sommerfeld boundary conditions (popular for BSSN formulation), for spherical boundary of radius $r = R$,

$$(\partial_t + \partial_r + \tfrac{1}{R})\psi_{ab} \doteq 0$$

- Spatial compactification [Pretorius 2005]
  - Choose mapping $r \to x(r)$ that maps spatial infinity to a finite coordinate location, e.g. $x = \arctan r$
  - Discretize uniformly in x
  - Apply low-pass frequency filter to damp waves as they travel out

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Our test problem

[Ongoing work with Lee Lindblom and Mark Scheel]

- Background solution: Schwarzschild black hole (mass $M = 1$)
- Add outgoing quadrupole wave perturbation [Teukolsky 1982], amplitude 4×10^{-3} (odd-parity)
- Evolve on a spherical shell extending from $r = 1.9$ (just inside the horizon) out to
  - $R = 1000$ (*reference solution*)
  - $R = 41.9, 81.9, \ldots$
- On the smaller domain, either impose the boundary conditions described in this talk or apply one of the alternative methods
- Compute difference of the two numerical solutions, compare in- and outgoing radiation (Ψ_0 and Ψ_4), ...

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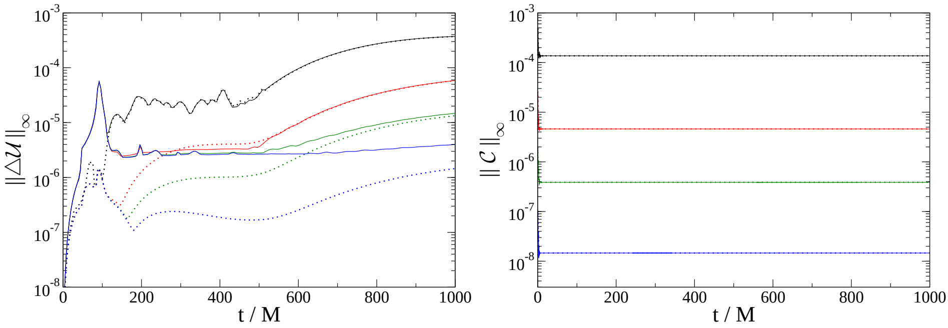
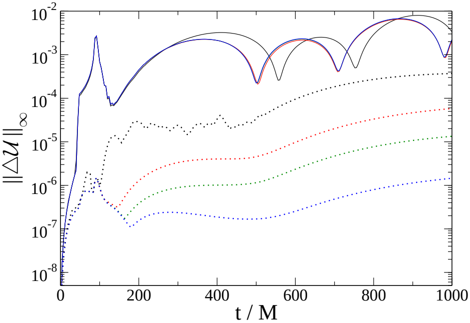
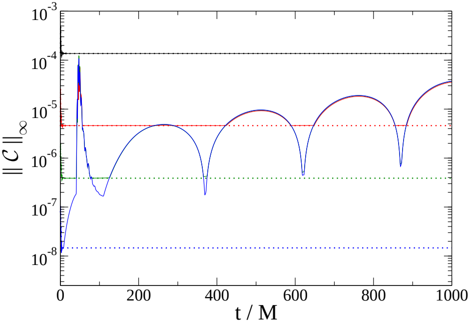
Old * (solid) vs. new (dotted) CPBCs
(* without the $\gamma_2\psi$ term in the gauge BCs)



$$R = 41.9,\ (N_r, L) = (21, 8), (31, 10), (41, 12), (51, 41)$$

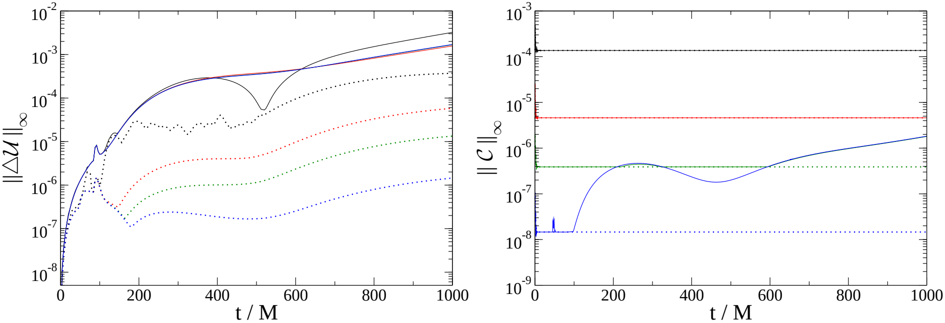
Freezing (solid) vs. new CP (dotted) BCs





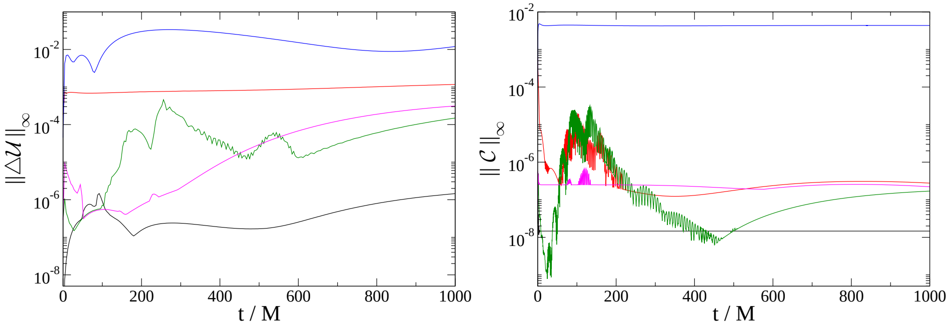
$R = 41.9$, $(N_r, L) = (21, 8), (31, 10), (41, 12), (51, 41)$

Sommerfeld (solid) vs. new CP (dotted) BCs



$R = 41.9$, $(N_r, L) = (21, 8), (31, 10), (41, 12), (51, 41)$

Tan compactification with various filters vs. new CPBCs



New constraint-preserving BCs

Kreiss-Oliger filter ($\epsilon = 1$) applied to RHS

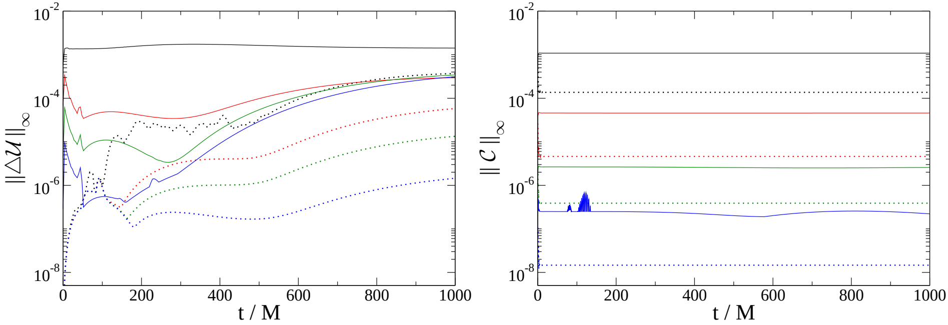
Hesthaven filter ($\sigma = 0.76, p = 13$) applied to RHS

Kreiss-Oliger filter ($\epsilon = 0.25$) applied to solution

Hesthaven filter ($\sigma = 0.76, p = 13$) applied to solution

$$R = 41.9, \ (N_r, L) = (51, 14)$$

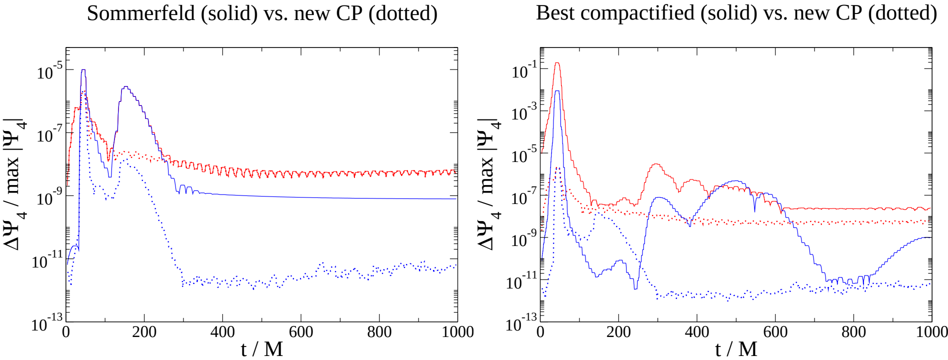
Tan compactification with best filter (solid) vs. new CPBCs (dotted)



Hesthaven filter applied to solution,

$R = 41.9$, $(N_r, L) = (21, 8), (31, 10), (41, 12), (51, 41)$

Accuracy of extracted Ψ_4

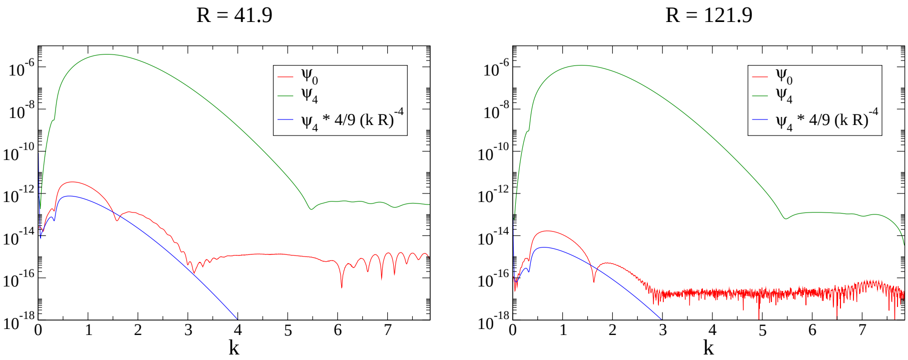


$$R = 41.9,\ (N_r, L) = (31, 10), (51, 41)$$

The reflection coefficient: theory vs. "experiment"

[Buchman & Sarbach 2006] predict for our CPBCs

$$\Psi_0/\Psi_4 = \frac{4}{9}(kR)^{-4} + O[(kR)^{-5}]$$



$$(N_r, L) = (51, 14)$$

Outline

Summary

- Constructed a set of constraint-preserving and radiation-controlling boundary conditions for the generalized harmonic Einstein equations
- Verified necessary conditions for well-posedness using the Fourier-Laplace technique, supported by numerical robust stability tests
- Numerical results indicate that our BCs cause significantly less reflections than alternate methods such as spatial compactification or Sommerfeld BCs

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Some references

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A new generalized harmonic evolution system
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[Kreiss & Winicour 2006] H.-O. Kreiss and J. Winicour
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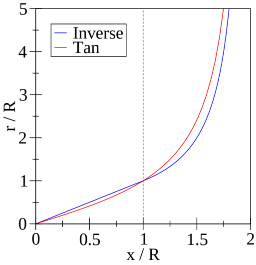
Numerical relativity using a generalized harmonic decomposition
Class. Quantum Grav. **22**(2) 425–451

Spatial compactification: details

Compactification map e.g.

- $r(x) = R\ \tan(\pi x/4R)$ (*Tan mapping*)
- $r(x) = \begin{cases} x, & 0 \leqslant x < R \\ R^2/(2R-x), & R \leqslant x < 2R \end{cases}$ (*Inverse mapping*)



Filter function e.g.

- $f(k) = 1 - \epsilon\ \sin^4(\pi k/2k_{max})$, where $0 \leqslant \epsilon \leqslant 1$ (*Kreiss-Oliger filter*)
- $f(k) = \exp[-(k/\sigma k_{max})^p]$, typically $\sigma = 0.76$, $p = 13$ (*Hesthaven filter*)