UNIQUENESS OF ILL POSED CHARACTERISTIC PROBLEMS FOR WAVE EQUATIONS

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MAIN RESULTS

Fact. Characteristic problem is not well posed in the complement of a domain of influence

$$\mathcal{D}(D_0) = \mathcal{J}^+(D_0) \cup \mathcal{J}^-(D_0).$$

One expects nevertheless that, under some reasonable conditions,

U.P. *Any smooth solutions of a wave equation in* $\mathcal{D}$ *which coincide on the boundary of* $\mathcal{D}(D_0)$ *must coincide in the complement of* $\mathcal{D}(D_0)$.

Results:

I. **U.P.** holds true in Minkowski space

II. **U.P.** holds true, near the horizon, in the domain of outer communication of a stationary black hole.

Hope:

III. Uniqueness of Kerr for non-analytic stationary solutions.

WAVE EQUATIONS IN $\mathbb{R}^{3+1}$

Consider,

$$\Box\Phi = F(\phi, \partial\phi) \tag{1}$$

in,

$$\mathcal{D} = \{(t, x) \in \mathbb{R}^{3+1}, \quad |x| > t + 1\}.$$

Theorem I. *Assume* $\phi_{(1)}, \phi_{(2)} \in \mathcal{C}^2(\overline{\mathcal{D}})$ *verify* (1) *in* $\mathcal{D}$ *and coincide on* $\delta(\mathcal{D})$. *Then,*

$$\phi_{(1)} \equiv \phi_{(2)} \qquad \textit{in} \quad \overline{\mathcal{D}}$$

Remark. It suffices to prove the result for linear equations with continuous coefficients in $\overline{\mathcal{D}}$,

$$F = A^\alpha \partial_\alpha \phi + B\phi$$

CARLEMAN ESTIMATE

Introduce $r = |x| = u + v + 1, \quad t = v - u$.

$$\begin{aligned} \mathcal{D} &= \{(u, v, \omega), \quad u, v \geq 0, \ \omega \in \mathbb{S}^2\} \\ \partial\mathcal{D} &= \{(u, v, \omega), \quad u, v \geq 0, \ uv = 0\} \end{aligned}$$

Introduce,

$$f(u, v) = \log(u + \frac{1}{4}) + \log(v + \frac{1}{4}) \qquad (2)$$

For $R \geq 1$,

$$\mathcal{D}_R = \{(u, v, \omega) \in \mathcal{D} : u + v + 1 < R\}.$$

Proposition 1. Given $R \geq 2$, there exist $C_R > 0$, $\beta(R) > 0$, such that for all $\phi \in \mathcal{C}_0^2(\mathcal{D}_R)$ and all $\beta \geq \beta(R)$, we have, with $\| \ \| = \| \ \|_{L^2}$,

$$\beta^{-\frac{1}{2}} \|e^{-\beta f} \Box \phi\| \geq C_R^{-1} \beta \|e^{-\beta f} \phi\| + \|e^{-\beta f} \partial \phi\|$$

Proposition 2. *Carleman Estimate* $\Rightarrow$ *Uniqueness*

PROOF OF CARLEMAN ESTIMATE

Set $\phi = e^{\beta f}\psi$ with $f = f(u, v)$ as above. Observe that,

$$\begin{aligned} e^{-\beta f}\Box(e^{\beta f}\psi) &= L\psi + \beta\Box f\psi \\ L\psi := L_{f,\beta}\psi &= \Box\psi + 2\beta\partial^\alpha f\partial_\alpha\psi + \beta^2(\partial_\alpha f\partial^\alpha f)\psi \\ &= \Box\psi + \beta V\psi + \beta^2(\partial_\alpha f\partial^\alpha f)\psi \end{aligned}$$

STEP 1. *It suffices to prove, with $C = C_R$,*

$$\beta\|\psi\| + C^{-1}\|\partial\psi\| \leq C\beta^{-1/2}\|L\psi\| \tag{3}$$

STEP 2. *It suffice to find g such that the quantity*

$$\begin{aligned} E[\psi] &:= < L\psi, \beta(V-g)\psi >_R \\ &:= \beta\int_{\mathcal{D}_R} L\psi\Big(V(\psi) - g\psi\Big) \end{aligned}$$

verifies the lower bound,

$$E \geq C_R^{-1}\Big(\beta\|\partial\psi\|^2 + \beta^3\|\psi\|^2\Big) + \beta^2\|(V-g)\psi\|^2 \tag{4}$$

Remark. Estimate (3) follows from (4) and the upper bound,

$$E \leq \beta\|L\psi\|_{L^2}\|(V-g)\psi\|_{L^2}$$

Recalling the definition of L,

$$
\begin{aligned}
E[\psi] &= \beta < L\psi, (V-g)\psi >_R \\
&= \beta^2 \|(V-g)\psi\|^2_{L^2} + E_1[\psi] + E_2[\psi] \\
E_1[\psi] &= +\beta < \Box\psi, (V-g)\psi > \\
E_2[\psi] &= < (\beta^3 f_u f_v + \beta^2 g)\psi, (V-g)\psi >
\end{aligned}
$$

Proposition 3. *To prove the Carleman estimate it suffices to show,*

$$
E_1[\psi] + E_2[\psi] \;\geq\; C_R^{-1}\Big(\beta\|\partial\psi\|^2 + \beta^3\|\psi\|^2\Big) \tag{5}
$$

for an appropriate choice of the functions f, g.

Proposition 4. *Estimate* (5) *is valid provided that the following inequalities hold true, uniformly in $\mathcal{D}_R$ with $h = g + \frac{1}{r}(f_u + f_v)$,*

$$
\begin{aligned}
C_R^{-1} &\leq f_{uv} + h - \frac{1}{r}(f_u + f_v) \\
C_R^{-1} &\leq A(u,v) := f_{uu} f_{vv} - h^2 \\
C_R^{-1} &\leq B(u,v) := -\frac{1}{2} f_u^2 f_{vv} - \frac{1}{2} f_v^2 f_{uu} - h f_u f_v
\end{aligned}
$$

PROOF OF PROPOSITION 4

SEP 3. For every ψ compactly supported in $\mathcal{D}_R$ we have,

$$E_1[\psi] = \beta < \Box\psi, (V-g)\psi >= \beta(J_1 + J_2 + J_3)$$

where,

$$\begin{aligned} J_1 &= \int_{\mathcal{D}_R} \Big(-\frac{1}{2} f_{vv} |\partial_u \psi|^2 - \frac{1}{2} f_{uu} |\partial_v \psi|^2 + h \partial_u \psi \partial_v \psi \Big) \\ J_2 &= \int_{\mathcal{D}_R} (f_{uv} + h - \frac{1}{r}(f_u + f_v)) |\nabla \psi|^2 \\ h &= g + \frac{1}{r}(f_u + f_v) \\ J_3 &= -\frac{1}{2} \int_{\mathcal{D}_R} \Box g \ \psi^2 = O(\|\psi\|^2) \end{aligned}$$

Proposition 5. *We have,*

$$E_1[\psi] \geq C_R^{-1} \beta \|\partial \psi\|^2 + O(\beta \|\psi\|^2)$$

provided that the following hold, uniformly in $\mathcal{D}_R$*,*

$$\begin{aligned} f_{uv} + h - \frac{1}{r}(f_u + f_v) &\geq C_R^{-1} \\ A(u,v) := f_{uu} f_{vv} - h^2 &\geq C_R^{-1} \end{aligned}$$

STEP 4. We establish a lower bound for $E_2[\psi]$.

$$\begin{aligned} E_2[\psi] &= < (\beta^3 f_u f_v + \beta^2 g)\psi, (V-g)\psi >_R \\ &= \beta^3 < f_u f_v \psi, (V-g)\psi >_R + O(\beta^2 \|\psi\|^2_{L^2}) \end{aligned}$$

Since $f_{uv} = 0$,

$$\begin{aligned} &< f_u f_v \psi, V\psi >_R = \int_{\mathcal{D}_R} f_u f_v \partial^\alpha f \partial_\alpha (\psi)^2 \\ &= -\int_{\mathcal{D}_R} \Big(\partial_\alpha (f_u f_v) \partial^\alpha f + (f_u f_v) \Box f \Big) \psi^2 \\ &= -\int_{\mathcal{D}_R} \Big(\frac{1}{2} f_u^2 f_{vv} + \frac{1}{2} f_v^2 f_{uu} + f_u f_v \Box f \Big) \psi^2 \end{aligned}$$

Since

$$g + \Box f = g + \frac{1}{r}(f_u + f_v) = h,$$

$$\begin{aligned} &< f_u f_v \psi, (V-g)\psi >_R \\ &= -\int_{\mathcal{D}_R} \Big(\frac{1}{2} f_u^2 f_{vv} + \frac{1}{2} f_v^2 f_{uu} + f_u f_v (g + \Box f \Big) \psi^2 \\ &= -\int_{\mathcal{D}_R} \Big(\frac{1}{2} f_u^2 f_{vv} + \frac{1}{2} f_v^2 f_{uu} + h f_u f_v \Big) \psi^2 \end{aligned}$$

The integrand is strictly positive if

$$C_R^{-1} \leq B(u,v) := -\frac{1}{2} f_u^2 f_{vv} - \frac{1}{2} f_v^2 f_{uu} - h f_u f_v$$

Choice of functions

Choose

$$\begin{aligned} f(u,v) &= \log(u+\frac{1}{4}) + \log(v+\frac{1}{4}), \\ h(u,v) &= \frac{u+v+\frac{3}{4}}{(u+\frac{1}{4})(v+\frac{1}{4})(u+v+1)} \end{aligned}$$

Then,

$$\begin{aligned} g &= \frac{1}{4(u+\frac{1}{4})(v+\frac{1}{4})(u+v+1)} \\ A &= \frac{1}{(u+\frac{1}{4})^2(v+\frac{1}{4})^2}\Big(1 - \frac{(u+v+\frac{3}{4})^2}{(u+v+1)^2}\Big) \\ B &= \frac{1}{4(u+\frac{1}{4})^2(v+\frac{1}{4})^2(u+v+1)} \end{aligned}$$

with,

$$\begin{aligned} A(u,v) &= f_{uu}f_{vv} - h^2 \\ B(u,v) &= -\frac{1}{2}f_u^2 f_{vv} - \frac{1}{2}f_v^2 f_{uu} - h f_u f_v \end{aligned}$$

II. UNIQUENESS PROPERTIES IN CURVED SPACETIMES

Conformal Property of Carleman Estimates

Let u, v define a double null foliation on $\mathcal{D} \subset \mathcal{M}$ with lapse,

$$g^{a\beta}\partial_\alpha u \partial_\beta v = \frac{1}{2\Omega^2}$$

We look for inequalities,

$$\beta\|e^{-\beta f}\phi\| + C_{\mathcal{D}}\|e^{-\beta f}D\phi\| \leq C_{\mathcal{D}}\beta^{-1/2}\|e^{-\beta f}\Box_g\phi\| \quad (*)$$

with,

$$\|e^{-\beta f}D\phi\|^2 = \int_{\mathcal{D}} e^{-2\beta f}\Big(|e_3(\phi)|^2 + |e_4(\phi)|^2 + |\nabla\phi|^2\Big)$$

and e_3, e_4 an associated normalized null pair.

Proposition. *Assume that* $\log\Omega$ *and its frame derivatives are bounded in* $\mathcal{D}$. *Then, for sufficiently large* β, *to prove a Carleman estimate for* $\Box_g$ *it suffices to prove one for* $\Box_{g'}$ *with* $g' = \Omega^{-2}g$.

We may thus assume that $\Omega = 1$.

Further Reductions

Introducing $\phi = e^{\beta f}\psi$ it suffices to prove,

$$\beta\|\psi\|_{L^2} + C^{-1}\|D\psi\|_{L^2} \le C\beta^{-1/2}\|L\psi\|_{L^2}$$

where,

$$\begin{aligned} L\psi &= \Box_g\psi + \beta V\psi + \beta^2 D_\alpha f D^\alpha f\psi, \\ V &= 2D^\alpha f\partial_\alpha. \end{aligned}$$

It suffices to prove a lower bound for $E_1[\psi] + E_2[\psi]$ with,

$$\begin{aligned} E_1[\psi] &= \beta < \Box\psi, (V - g)\psi >_g \\ E_2[\psi] &= < (\beta^3 f_u f_v + \beta^2 g)\psi, (W - w)\psi >_g \end{aligned}$$

It suffices to prove,

$$E_1[\psi] + E_2[\psi] \ \ge \ C_R^{-1}\Big(\beta\|\partial\psi\|^2 + \beta^3\|\psi\|^2\Big)$$

for an appropriate choice of the functions f, g.

Proposition. *For every ψ compactly supported in $\mathcal{D}$ we have,*

$$E_1[\psi] = < \Box\psi, (W - w)\psi >_g = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4$$

where, setting,

$$h = w + \frac{1}{2}(tr\chi f_u + tr\underline{\chi} f_v) \qquad (6)$$

$$\begin{aligned}
\mathcal{J}_1 &= \int_{\mathcal{D}} \Big(-\frac{1}{2} f_{vv} |\psi_3|^2 - \frac{1}{2} f_{uu} |\psi_4|^2 + h\psi_3\psi_4 \Big) \\
\mathcal{J}_2 &= \int_{\mathcal{D}} \Big(f_{uv} + h - \frac{1}{2}(tr\chi f_u + tr\underline{\chi} f_v) \Big) |\nabla\psi|^2, \\
\mathcal{J}_3 &= -\int_{\mathcal{D}} \zeta \cdot \nabla\psi(-f_v\psi_3 + f_u\psi_4) \\
&- \int_{\mathcal{D}} \Big(\chi_{ab} f_u + \widehat{\underline{\chi}}_{ab} f_v) \psi_a \psi_b \\
\mathcal{J}_4 &= -\frac{1}{2} \int_{\mathcal{D}} (\Box w) \psi^2
\end{aligned}$$

Also,

$$\begin{aligned}
E_2 &= -\beta^3 \int_{\mathcal{D}} \Big(\frac{1}{2} f_{uu} f_v^2 + \frac{1}{2} f_{vv} f_u^2 + f_u f_v (2 f_{uv} + h) \Big) \psi^2 \\
&+ O(\beta^2 \|\psi\|^2_{L^2} \Big)
\end{aligned}$$

Proposition. *Assume that f and h verify,*

$$\begin{aligned} 0 &< f_{uu}f_{vv} - h^2 \\ 0 &< f_{uv} + h - \frac{1}{2}(\mathit{tr}\chi f_u + \mathit{tr}\underline{\chi} f_v) \\ 0 &< -\frac{1}{2}f_{uu}f_v^2 - \frac{1}{2}f_{vv}f_u^2 - f_u f_v(2f_{uv} + h) \end{aligned}$$

Then,

$$E_1[\psi] + E_2[\psi] \geq C_R^{-1}(\beta\|\nabla\psi\|^2 + \beta^3\|\psi\|^2) + \mathcal{J}_3$$

Proposition. *If*

$$\begin{aligned} f &= \log(u+\epsilon) + \log(v+\epsilon) \\ h &= \frac{1-\varepsilon_0}{(u+\epsilon)(v+\epsilon)} \end{aligned}$$

then the above inequalities are satisfied, as long as

$$v\mathit{tr}\chi + u\mathit{tr}\underline{\chi} < 2.$$