Trapping horizons in constrained evolutions of excised black holes

Eric Gourgoulhon and José Luis Jaramillo

Laboratoire de l'Univers et de ses Théories (LUTH)
Observatoire de Paris
92195 Meudon, France

Instituto de Astrofísica de Andalucía (IAA-CSIC)
Granada, Spain

Based on collaboration with:
initial data: Marcus Ansorg, Silvano Bonazzola, Sergio Dain, Badri Krishnan, François Limousin, Guillermo Mena-Marugán
evolution analysis: Isabel Cordero-Carrión, J.M. Ibáñez, Jérôme Novak, Nicolas Vasset

From geometry to numerics (*General Relativity Trimester IHP*),
Paris, 22 November 2006

Plan of the talk

1. **General Objective :** BH evolutions
2. **Methodology :**
   a) Dynamical trapping horizons
   b) Constrained evolution scheme
   c) Excision

   $\Longrightarrow$ *Specific objective* : BH inner boundary conditions
3. **Antecedents :** Isolated horizons and BH Initial data
4. **Results :** Boundary conditions for dynamical trapping horizons
5. **Conclusions**

General objective

Main goal

Evolution of black hole spacetimes, with emphasis in the study and control of the properties of the horizon

since Event Horizon global...

Local characterization of the black hole horizon by means of the *dynamical* and *trapping horizons*, based on the notion of marginally trapped surface ("apparent horizons"...)

...**a priori** vs. **a posteriori** analysis (cf. talk by B. Krishnan)

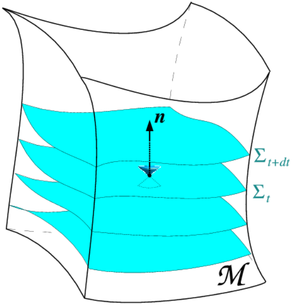
(Dreyer et al. 03, Baiotti et al. 05, Schnetter et al. 06, Schnetter & Krishnan 06...)

Motivations

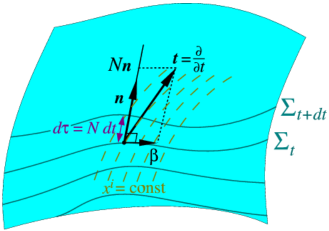
- Binary black hole evolutions and Gravitational Waves
- Geometrical properties of trapping horizons

Methodology

Geometric inner boundary conditions : 3+1 notation

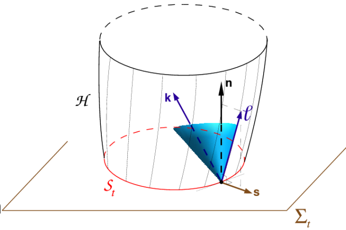


$\{\Sigma_t\}$	3+1 slicing of spacetime
n^μ	timelike unit normal to Σ_t
$t^\mu = N n^\mu + \beta^\mu$	evolution vector
N	lapse function
β^μ	shift vector
$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$	spatial 3-metric
$K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu}$	extrinsic curvature

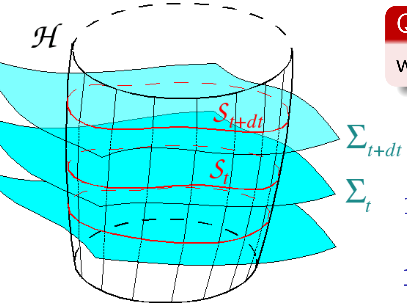


Dynamical trapping Horizons (see previous talks by S. Hayward and B. Krishnan)

based on (marginally) trapped surfaces $\mathcal{S}_t$



s^μ	unit normal vector to $\mathcal{S}_t$, in Σ_t
ℓ^μ	outgoing null vector
k^μ	ingoing null vector, $k^\mu \ell_\mu = -1$
$q_{\mu\nu} = \gamma_{\mu\nu} - s_\mu s_\nu$	induced metric on $\mathcal{S}_t$
$\theta^{(\boldsymbol{\ell})} \equiv q^{\mu\nu}\nabla_\mu \ell_\nu = 0$	Vanishing (outgoing) expansion
$\theta^{(\boldsymbol{k})} \equiv q^{\mu\nu}\nabla_\mu k_\nu < 0$	Negative (ingoing) condition



Quasi-local horizon

world-tube $\mathcal{H}$ of "apparent horizons", i.e. :

1. $\mathcal{H} \approx S^2 \times \mathbb{R}$ foliated by future marginally trapped 2-surfaces ($\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0$)

1.a **Future outer trapping horizon (FOTH)** (Hayward 1994) : $\mathcal{L}_{\boldsymbol{k}}\, \theta^{(\boldsymbol{\ell})} < 0$

1.b **Dynamical horizon (DH)** (Ashtekar & Krishnan 2002) : $\mathcal{H}$ spacelike.

Characterizations of the equilibrium situation

- **Non-expanding horizons (NEH)** (Hájíček 1973) : Null limit of a FOTH (Hayward ; Booth & Fairhurst 06) $\Longrightarrow$ intrinsic geometry invariant : $\mathcal{L}_\ell \boldsymbol{q} = 0$.
  - $\mathcal{H}$ null hypersurface (ℓ null normal)
  - $\theta_{(\ell)} = 0$
- **Isolated Horizons (IH)** (Ashtekar et al. 1999) :
  - NEH, $-T_{\alpha\mu}\ell^\mu$ future directed
  - *Extrinsic geometry* (induced connection) invariant : $[\mathcal{L}_\ell, \hat{\nabla}] = 0$

Intermediate level : **Weakly Isolated Horizons (WIH)**

- NEH
- $\hat{\nabla}$ *vertical* components, $\hat{\nabla}_\alpha \ell^\beta = \omega_\alpha \ell^\beta$, invariant :

$$\mathcal{L}_\ell \boldsymbol{\omega} = 0 \quad \Leftrightarrow \quad \boxed{\hat{\nabla}\kappa_{(\ell)} = 0} \quad , \quad \text{with } \kappa_{(\ell)} = \omega_\mu \ell^\mu$$

- **Slowly Evolving Horizons (SEH)** (Booth & Fairhurst 2004) : Geometrical characterization of a FOTH near equilibrium

Fully-constrained evolution scheme I : conformal decomposition

Conformal decomposition of (γ_{ij}, K^{ij}) on Σ_t :

- 3-metric

$$\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij} \quad , \quad \tilde{\gamma}^{ij} := f^{ij} + h^{ij}$$

  with $\tilde{\gamma}_{ij}$ unimodular : $\det(\tilde{\gamma}_{ij}) = \det(f_{ij})$ (f_{ij} background flat metric)

- Extrinsic curvature

$$K_{ij} = \Psi^{\zeta} \tilde{A}_{ij} + \frac{1}{3} K \gamma_{ij}$$

  where

$$\tilde{A}^{ij} = \frac{\Psi^{4-\zeta}}{2N} \left(\tilde{D}^i \beta^j + \tilde{D}^j \beta^i - \frac{2}{3} \tilde{D}_k \beta^k \tilde{\gamma}^{ij} + \dot{\tilde{\gamma}}^{ij} \right)$$

Fully-constrained evolution scheme II : Equations

Constraints + trace of evolution equations (with a choice for K and $\dot{K}$)

$$\begin{aligned}
\tilde{D}_k \tilde{D}^k \Psi - \frac{{}^3\tilde{R}}{8}\Psi &= S_\Psi[\Psi, N, \beta^i, K, \tilde{\gamma}, ...] \\
\tilde{D}_k \tilde{D}^k \beta^i + \frac{1}{3}\tilde{D}^i \tilde{D}_k \beta^k + {}^3\tilde{R}^i{}_k \beta^k &= S_\beta[\Psi, N, \beta^i, K, \tilde{\gamma}, ...] \\
\tilde{D}_k \tilde{D}^k N + 2\tilde{D}_k \ln\Psi\, \tilde{D}^k N &= S_N[N, \Psi, \beta^i, K, \tilde{\gamma}, \dot{K}, ...]
\end{aligned}$$

Evolution equations (trace-less part) + generalized Dirac's gauge : $\mathcal{D}_k \tilde{\gamma}^{ki} = 0$ (Bonazzola et al. 04)

$$\frac{\partial^2 \tilde{\gamma}^{ij}}{\partial t^2} - \frac{N^2}{\Psi^4}\Delta\tilde{\gamma}^{ij} - 2\mathcal{L}_{\boldsymbol{\beta}}\frac{\tilde{\gamma}^{ij}}{\partial t} + \mathcal{L}_{\boldsymbol{\beta}}\mathcal{L}_{\boldsymbol{\beta}}\tilde{\gamma}^{ij} = S^{ij}_{\tilde{\gamma}}[N, \Psi, \beta^i, K, \tilde{\gamma}, ...]$$

Elliptic part : re-scaled coupled PDE (cf. talk by D. Walsh)

Rescaling : $N = \tilde{N}\psi^a$

- $$\tilde{\Delta}\Psi - \frac{\tilde{R}}{8}\Psi + \frac{1}{32}\Psi^{5-2a}\tilde{N}^{-2}(\tilde{L}\beta)_{ij}(\tilde{L}\beta)^{ij} - \frac{1}{12}K^2\Psi^5 = 0,$$

- $$\tilde{\Delta}\beta^i + \frac{1}{3}\tilde{D}^i\tilde{D}_k\beta^k + \tilde{R}^i_k\beta^k - \tilde{N}^{-1}(\tilde{L}\beta)^{ik}\tilde{D}_k\tilde{N}$$
  $$-(a-6)\Psi^{-1}(\tilde{L}\beta)^{ik}\tilde{D}_k\Psi = \frac{4}{3}\Psi^a\tilde{N}\tilde{D}^iK \quad ,$$

- $$\tilde{\Delta}\tilde{N} + 2(a+1)\tilde{D}^k\ln\Psi\tilde{D}_k\ln\tilde{N}$$
  $$+\tilde{N}\left[\frac{a}{8}\tilde{R} + \frac{a-4}{12}\Psi^4K^2 + a(a+1)\tilde{D}^k\ln\Psi\tilde{D}_k\ln\Psi\right]$$
  $$-\frac{a+8}{32}\Psi^{4-2a}\tilde{N}^{-1}(\tilde{L}\beta)_{ij}(\tilde{L}\beta)^{ij} = \Psi^{4-a}\beta^k\tilde{D}_kK \quad .$$

No obvious (...possible ?) choice of a for applying a maximum principle...

Specific problem : Excision Method and Inner Boundary Conditions

Excision method

We remove a sphere $\mathcal{S}_t$ for the integration domain Σ_t.
We enforce this surface to coincide with a spatial slice of the horizon $\mathcal{H}$.

Then...
Inner Boundary conditions :

Elliptic part : Ψ, $\beta^\perp$, V^i, N

- Hamiltonian constraint : Ψ.
- Momentum constraint : $\boldsymbol{\beta} = \beta^\perp \boldsymbol{s} - \boldsymbol{V}$, with $\beta^\perp = \beta^i s_i$ and $V^i s_i = 0$.
- Prescription of $\dot{K}$: N.

Evolution (hyperbolic) part : $\tilde{\gamma}^{ij}$ (see talk by J. Novak)

Study of the characteristics.

(Technical) Antecedents

Initial Data in instaneous quasi-equilibrium

Isolated Horizon boundary conditions

1) Marginally trapped surface — $\theta_{(\ell)} = 0$
2) Coordinate system adapted to the Horizon (t^μ tangent to $\mathcal{H}$) — $\beta^\perp = N$
3,4) Quasi-equilibrium condition — $\boldsymbol{\sigma}_{(\boldsymbol{\ell})} = \mathcal{L}_{\boldsymbol{\ell}} \boldsymbol{q} - \frac{1}{2}\theta_{(\boldsymbol{\ell})}\boldsymbol{q} = 0$

$$\left({}^2\tilde{D}_a \tilde{V}_b + {}^2\tilde{D}_b \tilde{V}_a - ({}^2\tilde{D}_c V^c)\,\tilde{q}_{ab} = 0\right)$$

5) A fifth BC (foliation...) — Freedom to choose slicing...

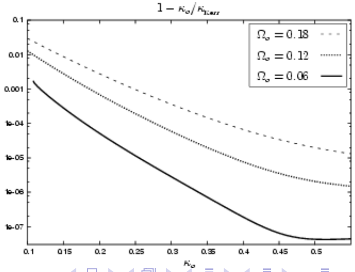
(Cook et al. 02,04, JLJ, Mena& Gourgoulhon 04)

Constant *surface gravity* prescription

$\kappa_{(\boldsymbol{\ell})} = \kappa_o = \mathrm{const}$ determines a unique solution of the CTS system, when combined with IH BCs : 1), 2), 3) and 4).

No solution if $\kappa_o = \kappa_{\mathrm{Kerr}}(a, J)$

(JLJ, Ansorg & Limousin 06)



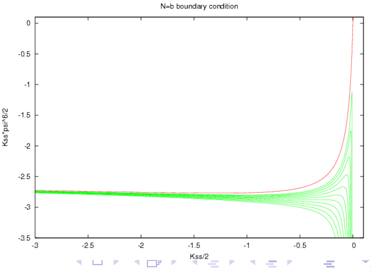
Prescription of the ingoing null normal expansion

Reasons for $\theta_{(k)} < 0$

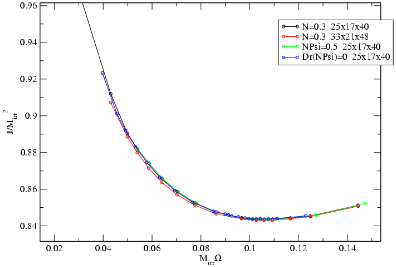
- $\mathcal{S}_t$ **future** marginally trapped surface
- **Increasing area** in the evolution of the marginally trapped surfaces
- **No self-intersections** in the evolution of $\mathcal{S}_t$ (Andersson, Mars & Simon 06)
- CTT isol. hor. initial data analysis (K=0), with $\hat{k}^\alpha = \frac{1}{2}(n^\alpha - s^\alpha)$:
  $\tilde{D}_i \tilde{s}^i \leq \Psi^6 \cdot \theta_{(\hat{k})} \leq 0 \Longrightarrow$ **existence and uniqueness** of the solution
  (Dain, JLJ & Krishnan 05)

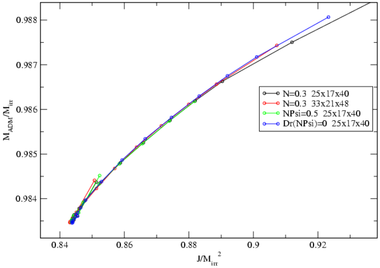
Prescription of $\theta_{(\hat{k})} = \mathrm{const} < 0$

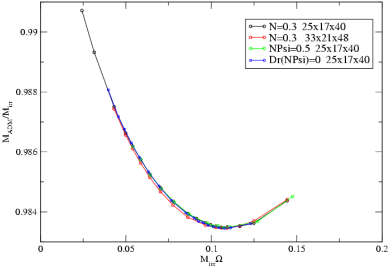
Existence and uniqueness of the solution to CTS equations (rescaled quantity $\Psi^6 \cdot \theta_{(\hat{k})}$ not a good parameter...)
(JLJ, Ansorg & Limousin 06)



Initial data of binary black holes (Cook et al. 04, Ansorg 05, Caudill et al. 06, Limousin & JLJ [in prep. ; extension of Grandclément et al. 02])



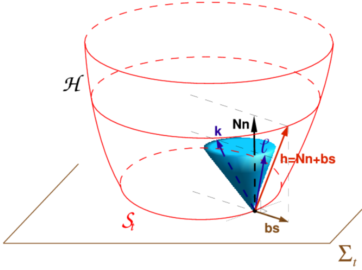




- Corotating BHs
- Maximal slicing, conformal flatness
- *Quasi-Killing* hellical vector : $\dot{\tilde{\gamma}} = 0$ and $M_{ADM} = M_{Komar}$

Inner Boundary conditions in the Dynamical case

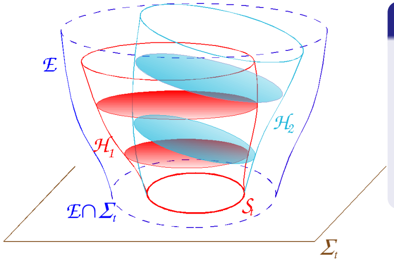
Uniqueness and existence results of dynamical trapping horizons



Result 1 (Ashtekar & Galloway 05)

Given a DH $\mathcal{H}$, the foliation by marginally trapped surfaces is unique

$\Rightarrow$ Unique vector $\boldsymbol{h}$: $\boldsymbol{h} = N\boldsymbol{n} + b\boldsymbol{s}$



Result 2 (Andersson, Mars & Simon 05)

Given a marginally trapped surface $\mathcal{S}_0$ in a Cauchy hypersurface Σ, to each 3+1 foliation $(\Sigma_t)_{t\in\mathbb{R}}$ it corresponds a unique DH $\mathcal{H}$ containing $\mathcal{S}_0$ and sliced by MTSs $\mathcal{S}_t \subset \Sigma_t$

Results 1 and 2 $\Rightarrow$ Non-uniqueness of the $\mathcal{S}_0$ evolution

Adapted coordinate system

Norm of $\boldsymbol{h}$ and type of $\mathcal{H}$

- Definition : $C := \frac{1}{2}\,\boldsymbol{h}\cdot\boldsymbol{h} = \frac{1}{2}\left(b^2 - N^2\right)$

$$\begin{array}{lllll} \boldsymbol{h} \text{ is spacelike} & \iff & C > 0 & \iff & \boxed{b - N > 0} \\ \boldsymbol{h} \text{ is null} & \iff & C = 0 & \iff & b - N = 0 \\ \boldsymbol{h} \text{ is timelike} & \iff & C < 0 & \iff & b - N < 0. \end{array}$$

In a FOTH $C \geq 0$ (Hayward), and sign of C is global on $\mathcal{S}_t$ (Booth & Fairhurst 06)

Coordinate system adapted to the horizon : $\boldsymbol{t}$ tangent to $\mathcal{H}$

Remember decomposition of the shift vector : $\boldsymbol{\beta} = \beta^{\perp}\boldsymbol{s} - \boldsymbol{V}$,

$$\boldsymbol{t} \text{ tangent to } \mathcal{H} \Leftrightarrow \boxed{\beta^{\perp} = b}$$

In this case, $\boldsymbol{h} = \boldsymbol{\partial}_t + \boldsymbol{V}$ and $\boxed{\beta^{\perp} - N > 0}$

Trapping horizon boundary conditions

- Apparent horizon condition $\theta_{(\ell)} = 0$ (Thornburg 87, Dain 04, Maxwell 04...) :

$$4\tilde{s}^i \tilde{D}_i \Psi + \tilde{D}_i \tilde{s}^i \Psi + \Psi^{-1} K_{ij} \tilde{s}^i \tilde{s}^j - \Psi^3 K = 0$$

- Definition of the trapping horizon : $\boldsymbol{\mathcal{L}_h}\, \theta^{(\boldsymbol{\ell})} = 0$ (Eardley 98...)

$$\left[-{}^2D_a {}^2D^a - 2L^a {}^2D_a + A \right] (\beta^\perp - N) = B(\beta^\perp + N)$$

with $L_a := K_{ij} s^i q^j{}_a$

$A := \frac{1}{2}\mathcal{R} - {}^2D_a L^a - L_a L^a - 4\pi T_{\mu\nu}(n^\mu + s^\mu)(n^\nu - s^\nu)$

$\mathcal{R}$: Ricci scalar of the metric $\boldsymbol{q}$ on $\mathcal{S}_t$

$B := \frac{1}{2}\sigma^{(\hat{\ell})}_{ab}\sigma^{(\hat{\ell})ab} + 4\pi T_{\mu\nu} \underbrace{(n^\mu + s^\mu)}_{:=\hat{\ell}} (n^\nu + s^\nu)$

Gauge boundary conditions I : *Tangential part of the shift*

Traceless part of deformation tensor along $\boldsymbol{h}$, $\boldsymbol{\sigma}^{(\boldsymbol{h})}$:

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{q} =: \theta^{(\boldsymbol{h})}\boldsymbol{q} + 2\boldsymbol{\sigma}^{(\boldsymbol{h})}$$

In adapted coordinates ($\boldsymbol{h} = \boldsymbol{\partial}_t + \boldsymbol{V}$) :

$$2\sigma^{(\boldsymbol{h})}_{ab} = \left(\frac{\partial q_{ab}}{\partial t} - \frac{\partial}{\partial t}\ln\sqrt{q}\; q_{ab}\right) + \left({}^2D_a V_b + {}^2D_b V_a - {}^2D_c V^c\, q_{ab}\right)$$

Coordinate choice

$$\boxed{\frac{\partial q_{ab}}{\partial t} - \frac{\partial}{\partial t}\ln\sqrt{q}\; q_{ab} = 0}$$

That is,

$$\boxed{{}^2D_a V_b + {}^2D_b V_a - {}^2D_c V^c\, q_{ab} = 2\sigma^{(\boldsymbol{h})}_{ab}}$$

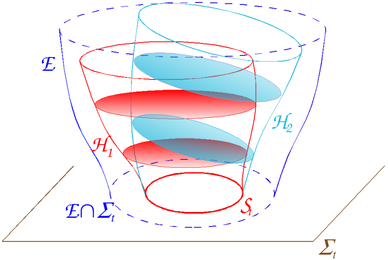
...where $\sigma^{(\boldsymbol{h})}_{ab}$ is determined via the evolution equation :

$$\begin{aligned}\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{\sigma}^{(\boldsymbol{h})} &= -\vec{\boldsymbol{q}}^*\mathbf{Weyl}(\underline{\boldsymbol{\ell}}, ., \boldsymbol{\ell}, .) - C^2\vec{\boldsymbol{q}}^*\mathbf{Weyl}(\underline{\boldsymbol{k}}, ., \boldsymbol{k}, .)\\ &- 8\pi C\left[\vec{\boldsymbol{q}}^*\boldsymbol{T} - \frac{1}{2}(\boldsymbol{q} : \boldsymbol{T})\boldsymbol{q}\right] + \cdots\end{aligned}$$

Gauge boundary conditions II : choice of DH

Remember non-uniqueness of the evolution of $\mathcal{S}$:

choice of foliation ⇔ choice of $\mathcal{H}$



Optimal geometrical choice of $\mathcal{H}$? (Gourgoulhon & JLJ 06)

- Maximization of the area increase rate $\dot{A}$: $b - N = -\underbrace{\mathrm{const}}_{>0} \cdot \theta^{(\hat{\boldsymbol{k}})}$
- Control of the convexity of the *area* function in time, $\ddot{A}$. Possibility of a smoother matching with an Isolated Horizon...

Viscous fluid analogy, entropy principles and fixing of $\mathcal{H}$

As in *Membrane Paradigm* (Damour79,82 ; Price & Thorne 86) for Event Horizons...

A dynamical horizon admits an analogy as a two-dimensional viscous fluid

System of Balance Equations on $\mathcal{H}$

Einstein equations on $\mathcal{H}$ ($\boldsymbol{m}$: time-like normal to $\mathcal{H}$, $\boldsymbol{m} \cdot \boldsymbol{m} = -2C$) :

- Component $\boldsymbol{T}(\boldsymbol{m}, \vec{\boldsymbol{q}})$: density of angular momentum J balance equation (Navier-Stokes-like equation)(Gourgoulhon 05)
- Component $\boldsymbol{T}(\boldsymbol{m}, \boldsymbol{h})$: *energy density* $\varepsilon := \theta^{(\boldsymbol{h})}/8\pi$ balance equation (Gourgoulhon & JLJ 06)

Balance equation for the *entropy*... ?

(Clausius-Duhem inequality of Non-Equilibrium Thermodynamics)

Tempting possibility : to base the choice of DH upon an **entropy principle** derived solely from the structure of the hyperbolic system defined by (part of) the Einstein equations on $\mathcal{H}$

First analysis of the characteristics of the hyperbolic part

(Cordero-Carrión et al.)

- Evolution equations on $\tilde{\gamma}_{ij}$ written as a first order system :
$$\partial_t \boldsymbol{U} + A^i(\boldsymbol{U})\partial_i \boldsymbol{U} = \boldsymbol{F}[\boldsymbol{U}, ...]$$
- Dirac gauge $\Rightarrow$ real characteristics : hyperbolic system
- Given the space-like vector $\boldsymbol{s}$ normal to $\mathcal{S}$, the associated characterictics (cf. e.g. talks by J. Winicour and O.Rinne) :

$$\begin{aligned} \lambda_0^{(\boldsymbol{s})} &= 0 \\ \lambda_\pm^{(\boldsymbol{s})} &= -\beta^\perp \pm N \end{aligned}$$

No ingoing characteristics if $-\beta^\perp + N \leq 0$, as it is the case if $\beta^\perp = b$ (adapted coordinate system), since $b - N \geq 0$.

Consequence

No need (no right !) to impose inner boundary conditions in the hyperbolic part, as a consequence of the BCs enforced in the elliptic part.

Conclusions I

Results

Derivation of a set of **inner boundary conditions** for the evolution of black hole space-times (*"from geometry to numerics..."*) :

a) From geometrical **dynamical trapping horizon** framework
b) In the context of a **fully-constrained** evolution scheme
c) Using an **excision** approach to BH singularities

Caveats !

- Potential problems coming from non-uniqueness in the elliptic system (cf. talk by Walsh) (Pfeiffer & York 05, Baumgarte et al. 06, Walsh 06)
  - Drop N equation coming from prescription of $\dot{K}$ and make an *appropriate* choice for $\partial_t N$... ?
  - Role of excision inner BCs for improving uniqueness issues ?
- In the more general case, the world-tube $\mathcal{H}$ will not be a FOTH during all the evolution.
  Example : Merger in BBH evolution and apparent horizon *jumps*.
  $\mathcal{L}_{\boldsymbol{k}}\theta^{(\boldsymbol{\ell})} \geq 0$ and/or $\theta_{(k)} \geq 0$ to allow for topology change

Conclusions II

Future directions

Preliminary analysis !

Urgent need to test in numerical implementations :

- Application to the BBH evolution problem (see tomorrow's session...)
- **Feedback on Geometry** (*"from numerics to geometry... ?"*) : Test of geometrical ideas in trapping horizons (e.g. asymptotic properties of these horizons, boundary of the spacetime *trapped region*... (e.g. Schnetter & Krishnan 06, Ben-Dov 06)).