Black Hole Rigidity in General Dimensions

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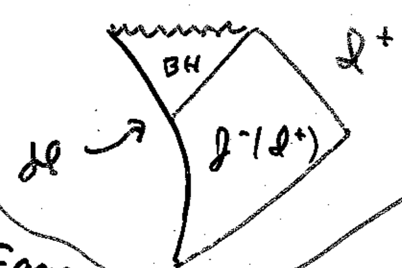
Model Black Hole Rigidity Thm

- let (M^{n+1}, g) be a stationary (not static) Black Hole

There exists a killing field T which is
- timelike asymptotically
- not surface forming

(M^{n+1}, g) is asympt flat with $\mathscr{I}^+$ & $M^{n+1} \setminus \mathscr{I}^-(\mathscr{I}^+) \neq$ empty



satisfying the Einstein-(?) Eqns together with the conditions $\begin{cases} \bullet \\ \bullet \\ \bullet \\ \bullet \end{cases}$

- The isometry group $\mathscr{G}(M^{n+1}, g)$ includes a 2 Dim Abelian subgroup which $\begin{cases} \bullet \\ \bullet \\ \bullet \\ \bullet \end{cases}$

Importance of BH Rigidity

- Step towards uniqueness

ex 3+1

BH Rigidity $\longrightarrow$ Axisym

$\Longrightarrow$ Kerr

- Surface Gravity:

If $\mathcal{G}(M^{n+1}, g)$ includes generators of $\mathcal{H}$ then

$$\left(\nabla_Y Y - K\, Y\right)\Big|_{\mathcal{H}} = 0$$

defines surface Gravity

$\Longrightarrow$ BH Thermo

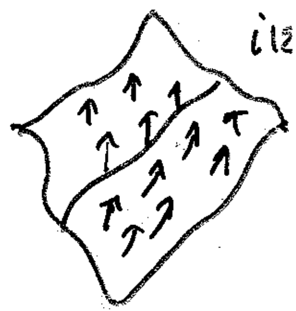
a real theorem:

3+1 Black Hole Rigidity (Hawking - Ellis, Chrusciel)

- Let (M^{3+1}, g) be a stationary (not static) BH satisfying the vacuum Einstein eqns or the Einstein - Maxwell eqns

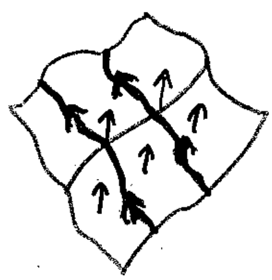
together with the conditions

- analytic
- $\mathcal{H} = \Sigma^2 \times \mathbb{R}$ — connected component; Σ^2 compact
- there exists $i : \Sigma^2 \to \mathcal{H}$ such that $i(\Sigma^2)$ spacelike, T transverse to $i(\Sigma^2)$



- The isometry group $\mathcal{G}(M^{3+1}, g)$ includes $S^1 \times \mathbb{R}$ subgroup

which acts freely on the **generators** of $\mathcal{H}$



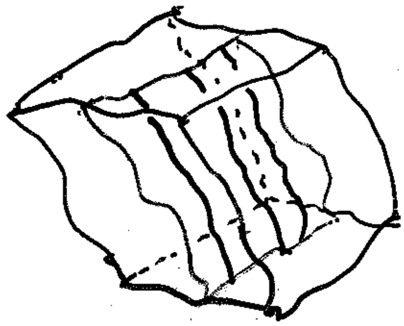
So there is a killing field <u>tangent to generators</u>

Proof Sketch of 3+1 BH Rigidity Thm

following ideas of Chruściel

A] Transformation into Compact Horizon Problem

- Consider neighborhood $\mathcal{U}$ of BH Horizon $\mathcal{H}$



- Recall

Thm (Hawking-Ellis)

$\mathcal{H} = \partial(J^-(\mathcal{I}^+))$ is

- 3 dim null hypersurface
- ruled by congruence of null generators
- $\Sigma^2 \times \mathbb{R} = S^2 \times \mathbb{R}$

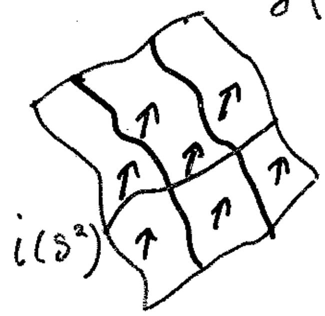
presuming that Σ^2 is compact

- Note that

T (Killing field of stationarity) is

- tangent to $\mathcal{H}$
  (since T preserves geometry)
- not tangent to generators of $\mathcal{H}$
  (Thm of Sudarsky-Wald
  relies on
  - stationary not static
  - asympt timelike

• transverse to $i(S^2)$



- Compactification

  Claim

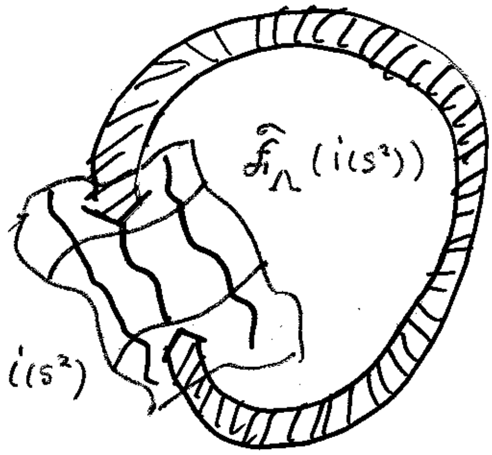
  Let $\hat{\mathcal{F}}_\lambda$ be flow of T

  $\exists \Lambda$ s.t.

  if identify

  $\hat{\mathcal{F}}_\Lambda(i(S^2)) \leftrightarrow i(S^2)$

  then generators <u>close</u>



  Pf of Claim

  Since flow on S^2 has fixed points

  $\hat{\mathcal{F}}_\lambda$ is λ-angle rotation

  so

  $\lambda = \Lambda \longleftrightarrow$ rotate by 2π !

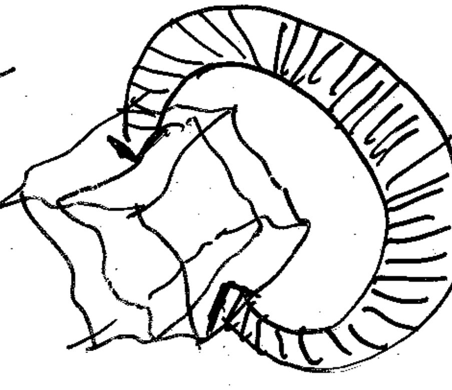
- Extend identification into $\mathcal{U}$

  $\Rightarrow$ New spacetime

  $(\hat{M}^{3+1}, \hat{g})$

  with <u>compact</u>

  null hypersurface $\hat{\mathcal{H}}$ "$S^2 \times S^1$"



ready to

B] Apply Compact Null Hypersurface Thm

- Thm (Moncrief-I)

  Let (M^{3+1}, g) be

  - soln of vac Eintein or Einstein-Maxwell
  - analytic

  Assume $\exists$

  $$N^3 \hookrightarrow M^{3+1}$$

  - analytic embedded 3 submfd
  - null
  - compact
  - closed generators

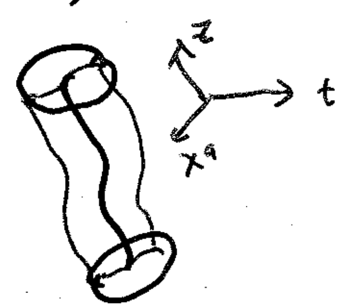
  The isometry group $\mathcal{G}(M^{3+1}, g)$ contains S^1 subgroup acting tangent to generators of N

- Idea of Proof
  - Set up Gaussian Null coords in tubes

  $$g = 2\,dt\,dz + \phi\,dz^2 + 2\beta_a\,dx^a dz + \mu_{ab}\,dx^a dx^b$$

  with $\phi|_N = 0$

  $\beta|_N = 0$



  - Show

  $\mu_{ab}|_N$ independent of z

pf

Hawking Ellis argument based on Raychaudhuri Shear Eq

$\Rightarrow g|_{\mathcal{H}}$ independent of z.

- Show $\exists$ new set of Gaussian Null Coords with $\partial_t \phi|_{\mathcal{H}} = K$ <u>constant</u>

  <u>pf</u>

  Solving Ricatti Eqn

  Note: This is where (t, z, x^a) are adjust so that $\frac{\partial}{\partial z}$ is KV

  Verify:
  $$\frac{d^2 z}{d\lambda^2} - \frac{K}{2}\left(\frac{dz}{d\lambda}\right)^2 = 0$$ geodesic eqn for generators

  $\Rightarrow K \underline{\text{is}}$ surface gravity

  $=0$ generators complete to future & pa

  $\neq 0$ generators complete only to future

- Show
  $$\partial_t^{(m)}\phi,\ \partial_t^{(m)}\beta_a,\ (\partial_t)^m \mu_{ab}$$
  all indep of z

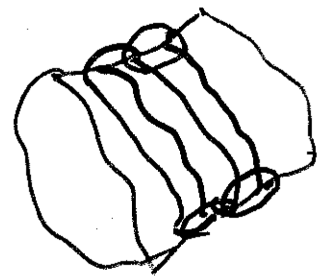
  <u>pf</u>

  Induction using Einstein Eqs

- Extend $\frac{\partial}{\partial z}$ as Killing field into Tube (off $\mathcal{H}$)

  Use Cauchy-Kowaleskaya ← Analyticity needed

• Patch Killing Vector Field from Tubes



C] UnWrap Back to Black Hole

¿ What Changes from 3+1 BH Rigidity Proof
to
n+1 BH Rigidity Proof

- Can still compactify $\mathcal{H}$

- But cannot compactify (generally)
with closed generators

$\Rightarrow$ n+1 version of Compact Null Hypersurface Thm
does not do the job

$\Rrightarrow$ Need
Compact Null Hypersurface Thm
with Non Closed Generators

Just developed by Moncrief - I
if have T symmetry

$n+1$ Black Hole Rigidity (Moncrief - I / Hollands - Ishibashi - Wald)

Let (M^{n+1}, g) $n \geq 3$ be a stationary (not static) BH

satisfying vacuum Einstein

together with the conditions

- analytic
- $\mathcal{H} = \Sigma^{n-1} \times \mathbb{R}$ compact
- there exists
  $i : \Sigma^{n-1} \to \mathcal{H}$
  such that $i(\Sigma^{n-1})$ spacelike
  T transverse to $i(\Sigma^{n-1})$
- generators complete to past
  incomplete to future ← New

The isometry group $\mathcal{G}(M^{n+1}, g)$

includes 2 dim subgroup

acting freely on the generators

Proof Sketch of $n+1$ BH Rigidity Thm

A] Transformation into Compact Horizon Problem

Same, except

Σ^{n-1} generally not S^2

$\longrightarrow$ generally no fixed pts for $\tilde{\mathcal{F}}_\lambda$

$\Longrightarrow$ generally non closed generators

rely on

B] Non Closed Generator Compact Null Hypersurface (with KV $\uparrow$) Thm

- Thm

Let (M^{n+1}, g) be

- soln of vac Einstein
- analytic

Assume

- $\exists\; N^n \hookrightarrow M^{n+1}$
  - * analytic embedding
  - * null
  - * compact $\Sigma^{n-1} \times S^1$
  - * non closed generators

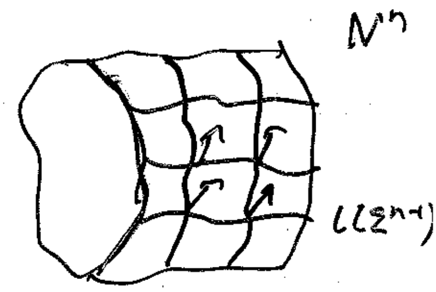
    generically

  - * generators complete to past, incomplete to future.

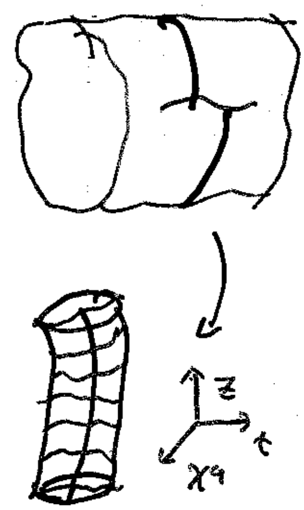
- $\exists$ Killing Vector field T
  transverse to some compact section $i(\Sigma^{n-1}) \subset N$

The isometry group $\mathcal{G}(M^{n+1}, g)$ contains
a T^2 subgroup acting tangent to generators of N^n

- Idea of Proof
  - Foliate N^n
    using $i(\Sigma^{n-1})$
    & flow of T
    $\to$ spacelike slices of N^n



  - Set up Gaussian Null Coordinates
    $\Rightarrow$ Same as for closed generators
    except at transition



$$g = 2\,dt\,dz + \phi\,dz^2 + 2\beta_a\,dx^a\,dz + \mu_{ab}\,dx^a dx^b$$

$$a, b \in \{1, \ldots n-1\}$$

with
$$\phi\big|_N = 0$$
$$\beta\big|_N = 0$$

• Show $\mu_{ab}|_N$ indep of z

Pf

As for 3+1

• Control Generator Orbits via

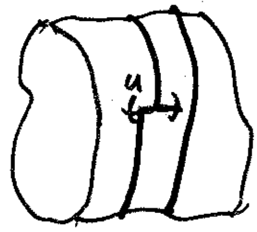
Poincaré Recurrence

~ Let $\{\psi_\lambda : M \to M \mid \lambda \in \mathbb{R}\}$

be 1-param family

of volume-preserving

diffeos of compact M



~ For any $p \in M$ & any $\lambda_0 > 0$ & any nbhd U of p

$\exists\ j \in \mathbb{N}$ s.t.

$$\psi_{j\lambda_0}(p) \in U$$

$\Rightarrow$ The orbit returns

arbitrarily close (eventually)

Note:

Preservation of

$\mu_{ab}\, dx^a dx^b$

geometry along

flow

$\Rightarrow T^k \times \mathbb{R}$ tubes as closure of flow

• Construct Candidate Killing Field

Let ∂_z be tangent to generators

Set $Y = w\,\partial_z$ on N in tube

$$w(z,x^a) = \frac{k}{2}\int_z^\infty d\rho\, \exp\left[-\int_z^\rho \tfrac{1}{2}\,\partial_t\phi(\xi,x^a)\,d\xi\right]$$

Integration along generator starting at (z,x^a)

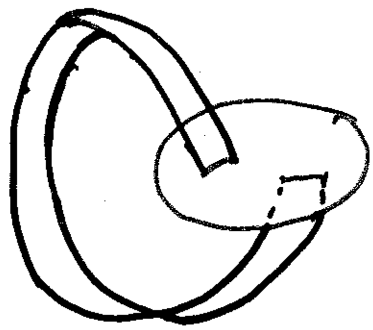
* Claim
  - For the null generator starting at (z,x^a) with tangent vector Y the future affine length is k regardless of (z,x^a)
  - The function w is analytic
  - The vector field Y extends to a Killing field

· Pf of Claim
  - uniform affine length → calc'n
  - analyticity is difficult
  - Killing field verification is just like closed generator case once analyticity is verified

★ Verifying Analyticity of $\mathcal{U}$

2 stages

* Continuity & Differentiability via Ribbon Arguments:



$\Rightarrow$ Compare integrals

$$\int -\tfrac{1}{2}\partial_t \phi$$

along neighboring gens starting at nearby points using Stokes on the ribbon with 1-form

$$\eta = -\tfrac{1}{2}\partial_t \phi$$

$$d\eta = 0$$

$$\Rightarrow \int \omega - \int \omega = \int \omega - \int \omega$$

* Analyticity via

- complexify
  "Grauert Tubes"
- construct sequence

$$\mathcal{U}_j(z, x^a) = \frac{k}{2}\int_z^{j\ t_{\text{recurrence}}} d\rho \,\exp[\text{ etc }]$$

in complex

↑ fol'n useful here

- show seq is Cauchy via Ribbons
- Use Banachness of $\mathbb{C}$ analytic

$\Rightarrow \mathcal{U} = \lim \mathcal{U}_j$ is analytic

• Verify Y is Killing field

  * Show

    $$\mathcal{L}_Y (\partial_t^{(m)} \phi) \qquad \mathcal{L}_Y (\partial_t^{(m)} \beta_a) \qquad \mathcal{L}_Y (\partial_t^{(m)} \mu_{ab})$$

    all vanish

  * Extend off N into tube via Cauchy-Kowalevskaya
  * Patch tubes

C] Unwrap Back to Black Hole

To Do:

- Complete write up
- Remove condition on generator completeness
- Add nonvacuum fields
- Remove analyticity
- Prove
  Non Closed Generator Compact Null Horizon Thm
  with no assumed Killing Field