Black-hole dynamics

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General Relativity, including Einstein equation, assumed throughout.

1. Classical theory of black holes

General Relativity formulated [Einstein 1915].

Gravitational field of a point mass M found [Schwarzschild 1916].

Charge Q added [Reissner 1916, Nordström 1918].

Wormhole spatial geometry understood [Einstein & Rosen 1935].

Maximally extended space-time geometry understood [Kruskal 1960, actually Wheeler].

Angular momentum J added [Kerr 1963, Newman et al. 1965].

Term "black hole" coined [Wheeler 1968].

Black holes defined by event horizons [Penrose 1968, Hawking & Ellis 1973].

"Second law": increase of area of event horizon, $A' \geq 0$ [Hawking 1971].

"Four laws of black-hole mechanics" formulated [Bardeen, Carter & Hawking 1973].

"Zeroth law": surface gravity κ constant on stationary black holes.

"First law": $\delta E = \kappa\delta A/8\pi + \Omega\delta J + \Phi\delta Q$ for perturbations of stationary black holes,

for ADM mass-energy E, angular speed Ω and electric potential Φ.

"Third law": $\kappa \not\to 0$, by perturbations of stationary black holes.

Note: this is black-hole statics, concerning Killing horizons, not general event horizons.

Are there corresponding laws of black-hole dynamics?

Are there conservation laws for energy and angular momentum?

This has become urgent, since black holes are now believed to exist in the universe,
as stellar-mass supernova remnants and as supermassive galactic cores,
and are expected to be major sources for current or planned gravitational-wave detectors,
e.g. by binary inspiral and merger, which the classical theory does not cover.

Event horizons, from which light can never escape, cannot be physically located by mortals.

In practice, a black hole consists of a region of trapped surfaces [Penrose 1965]
and can be located by a marginal surface (cf. apparent horizon),
where light is just trapped by the gravitational field.

2. Dynamical black holes [SAH 1994, 2006]

A spatial surface S in space-time has two unique future-pointing null normal directions,

along null normal vectors $l_\pm$: $g(l_\pm, l_\pm) = 0$, $\perp l_\pm = 0$,

where g is the space-time metric and $\perp$ is projection onto S.

The null expansions $\theta_\pm = L_\pm \log *1$,

where $*1$ is the area form of S, L the Lie derivative and $L_\pm = L_{l_\pm}$,

measure whether light rays are diverging, $\theta_+ > 0$, or converging, $\theta_- < 0$.

$$S \text{ is } \left\{ \begin{array}{c} \text{untrapped} \\ \text{marginal} \\ \text{trapped} \end{array} \right\} \text{ if } \left\{ \begin{array}{c} \theta_+\theta_- < 0 \\ \theta_+ = 0 \text{ or } \theta_- = 0 \\ \theta_+\theta_- > 0 \end{array} \right\} \text{ on } S, \text{ or equivalently, if } H \text{ is } \left\{ \begin{array}{c} \text{spatial} \\ \text{null} \\ \text{temporal} \end{array} \right\},$$

where the expansion vector or mean-curvature vector $H = g^{-1}(d \log *1)$,

$H = -e^f(\theta_- l_+ + \theta_+ l_-)$ where f is a normalization function, $e^{-f} = -g(l_+, l_-)$.

The expansion $\theta_\eta = L_\eta \log *1$ along any normal vector η, $\perp\eta = 0$, is $\theta_\eta = g(H, \eta)$.

Untrapped (or mean convex) surfaces have a local spatial orientation:

$$\text{an achronal (spatial or null) normal vector } \eta \text{ is } \left\{ \begin{array}{c} \text{outward} \\ \text{inward} \end{array} \right\} \text{ if } g(H,\eta) \left\{ \begin{array}{c} > 0 \\ < 0 \end{array} \right\}.$$

$$\text{Conventionally fix } \left\{ \begin{array}{c} \theta_+ > 0 \\ \theta_- < 0 \end{array} \right\} \text{ in an untrapped region; then } \left\{ \begin{array}{c} l_+ \text{ outward} \\ l_- \text{ inward} \end{array} \right\}.$$

Trapped surfaces have a local causal orientation:

$$\text{if } H \text{ is } \left\{ \begin{array}{c} \text{future} \\ \text{past} \end{array} \right\} \text{ causal (temporal or null), the surface is } \left\{ \begin{array}{c} \text{future} \\ \text{past} \end{array} \right\} \text{ trapped.}$$

$$\left\{ \begin{array}{c} \text{Future} \\ \text{Past} \end{array} \right\} \text{ trapped surfaces have } \theta_\pm \left\{ \begin{array}{c} < 0 \\ > 0 \end{array} \right\}.$$

Marginal surfaces have both orientations, if the other null expansion has fixed non-zero sign.

Trapping horizon: a hypersurface foliated by marginal surfaces,

$$\left\{ \begin{array}{c} \text{outer} \\ \text{inner} \end{array} \right\} \text{ if } L_\mp\theta_\pm \left\{ \begin{array}{c} < 0 \\ > 0 \end{array} \right\} \text{ for } \theta_\pm = 0, \text{ or equivalently, if } \nabla\cdot(H \mp H^*) \left\{ \begin{array}{c} > 0 \\ < 0 \end{array} \right\},$$

where $\eta^* = \eta^+ l_+ - \eta^- l_-$ is the normal dual vector to a normal vector $\eta = \eta^+ l_+ + \eta^- l_-$: $\perp\eta^* = 0$, $g(\eta^*, \eta) = 0$, $g(\eta^*, \eta^*) = -g(\eta, \eta)$.

$$\text{A } \left\{ \begin{array}{c} \text{future} \\ \text{past} \end{array} \right\} \text{ outer trapping horizon provides a local definition of a generic } \left\{ \begin{array}{c} \text{black} \\ \text{white} \end{array} \right\} \text{ hole.}$$

3. Basic laws [SAH 1994]

(i) Trapping: for a future/past, outer/inner trapping horizon,

there are trapped surfaces to one side and untrapped surfaces to the other side.

For a black-hole horizon, outgoing light rays are momentarily parallel, $\theta_+ = 0$,

diverging just outside, $\theta_+ > 0$, and converging just inside, $\theta_+ < 0$, since $L_-\theta_+ < 0$,

while ingoing light rays are converging, $\theta_- < 0$.

(ii) Signature:

$$\text{NEC (null energy condition)} \Rightarrow \left\{ \begin{matrix} \text{outer} \\ \text{inner} \end{matrix} \right\} \text{trapping horizons are} \left\{ \begin{matrix} \text{achronal} \\ \text{causal} \end{matrix} \right\},$$

and null if and only if the effective ingoing energy density (below) vanishes.

$\Rightarrow$ Black hole horizons are one-way traversable: one can fall into a black hole but not escape.

(iii) Area: NEC $\Rightarrow$

$$\left\{ \begin{matrix} \text{future outer or past inner} \\ \text{past outer or future inner} \end{matrix} \right\} \text{trapping horizons have} \left\{ \begin{matrix} \text{non-decreasing} \\ \text{non-increasing} \end{matrix} \right\} \text{area form,}$$

$$\left\{ \begin{matrix} \theta_\xi \geq 0 \\ \theta_\xi \leq 0 \end{matrix} \right\}, \text{and therefore area (if compact),} \left\{ \begin{matrix} L_\xi A \geq 0 \\ L_\xi A \leq 0 \end{matrix} \right\},$$

instantaneously constant ($\theta_\xi = 0$) if and only if the horizon is null,

where ξ is a normal generating vector of the marginal surfaces in the horizon

and $A = \oint_S *1$ is the area of the marginal surfaces.

$\Rightarrow$ Black holes grow if they absorb any matter or gravitational radiation,

and otherwise remain the same size.

(iv) Topology: DEC (dominant energy condition) $\Rightarrow$ a future/past outer trapping horizon

has marginal surfaces of spherical topology (if compact).

$\Rightarrow$ Realistic black holes are topologically spherical.

4. Conservation of energy [SAH 2004, cf. Ashtekar & Krishnan 2002]

Hawking mass [1968], simplest generalization of Schwarzschild $1 - 2M/r = g^{rr}$, units $G = 1$:

$$M = \frac{R}{2}\left(1 - \frac{1}{16\pi}\oint_S *g(H,H)\right) = \frac{R}{2}\left(1 + \frac{1}{8\pi}\oint_S *e^f\theta_+\theta_-\right), \text{ area radius } R = \sqrt{A/4\pi}.$$

Large spheres: $M \to \left\{\begin{matrix}\text{Bondi}\\ \text{ADM}\end{matrix}\right\}$ mass at $\left\{\begin{matrix}\text{null}\\ \text{spatial}\end{matrix}\right\}$ infinity in asymptotically flat space-times.

Small spheres: M/volume $\to$ density at a regular centre.

Trapping: a surface is $\left\{\begin{matrix}\text{trapped}\\ \text{marginal}\\ \text{untrapped}\end{matrix}\right\}$ if $\left\{\begin{matrix}R < 2M\\ R = 2M\\ R > 2M\end{matrix}\right\}$, i.e. M/R controls gravitational trapping.

Horizons: M is the irreducible mass of a future outer trapping horizon, $L_\xi M \geq 0$,

by the area law $L_\xi A \geq 0$, assuming NEC, since $A \cong 4\pi(2M)^2$,

where $\cong$ henceforth denotes evaluation on a marginal surface.

Recall the irreducible mass for stationary black holes:

$M \cong \sqrt{\frac{1}{2}m(m + \sqrt{m^2 - a^2})}$ for Kerr black holes [Christodoulou 1970],

the mass which must remain even if rotational energy is removed by the Penrose process.

Simplest generalization of Schwarzschild stationary Killing vector $k = \partial_t$?

Canonical time vector $k = (g^{-1}(dR))^* = e^f(L_+R\,l_- - L_-R\,l_+)$,

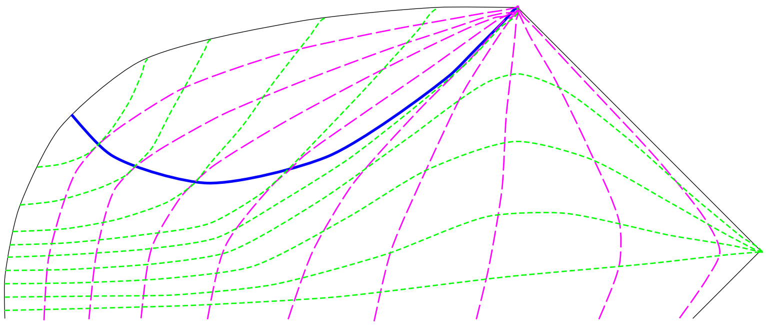
$\perp k = 0$, $k \cdot dR = 0$, $g(k,k) = -g^{-1}(dR, dR)$,

$g(k,k) \cong 0$, $k \cong \pm g^{-1}(dR)$ for $\theta_\pm \cong 0$,

$\Rightarrow$ trapping horizons can be defined by k being null,

cf. Killing horizons defined by stationary Killing vector being null,

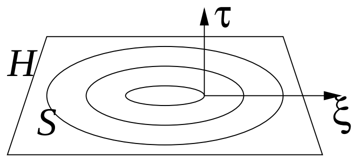
$\Rightarrow$ when a trapping horizon forms, the flow lines of k and $g^{-1}(dR)$ switch over.



Dual normal vector to horizon: $\tau = \xi^* = \xi^+ l_+ - \xi^- l_-$,

future-pointing for outward-pointing ξ,

$\tau \to \xi$ as a trapping horizon becomes null.

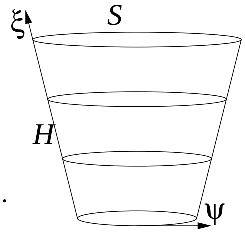


Conservation of energy takes the surface-integral form $L_\xi M \cong \oint_S *(T_{AB}+\Theta_{AB})k^A\tau^A$,
where T is the energy tensor of the matter
and Θ is the effective energy tensor of gravitational radiation (below).
Volume-integral form $[M] \cong \int_H *(T_{AB}+\Theta_{AB})k^A\tau^B \wedge dx$,
where x labels the marginal surfaces, $\xi = \partial_x$,
expresses the change $[M]$ in M between two marginal surfaces in the horizon H.
Proper-volume form $[M] \cong \int_H \hat{*}(T_{AB}+\Theta_{AB})k^A\hat{\tau}^B$ for a spatial trapping horizon,
with unit normal $\hat{\tau} = \tau/\sqrt{g_{xx}}$ and proper volume element $\hat{*}1 = *\sqrt{g_{xx}} \wedge dx$,
has the same form as the expression for energy with a stationary Killing vector k.
But $\hat{*}1 \to 0$, $\hat{\tau}$ ill defined if the trapping horizon becomes null, $g_{xx} \to 0$,
i.e. a growing black hole ceases to grow.

The energy conservation law expresses the increase in irreducible black-hole mass M
in terms of the energy densities of the infalling matter and gravitational radiation,
and therefore the increase in area $A \cong 4\pi(2M)^2$, so describes how a black hole grows.

Introduce the transverse metric h_{ab}, the induced metric of S,
the null shears $\sigma_{\pm ab} = \bot L_\pm h_{ab} - \theta_\pm h_{ab}$,
which are transverse ($\bot\sigma_{\pm ab} = \sigma_{\pm ab}$), traceless ($h^{ab}\sigma_{\pm ab} = 0$), symmetric bilinear forms,
and the normal fundamental forms $\zeta_{\pm a} = e^f \bot(l_{\pm\alpha}\nabla_a l^\alpha_\mp)$, which are transverse 1-forms.
Note indices α, $\beta\ldots$ are general, A, $B\ldots$ normal, a, $b\ldots$ transverse.
Then the components of Θ w.r.t. $l_\pm$ are $\Theta_{\pm\pm} = ||\sigma_\pm||^2/32\pi$, $\quad\Theta_{\pm\mp} = e^{-f}|\zeta_\pm|^2/8\pi$,
where $|\zeta|^2 = h^{ab}\zeta_a\zeta_b \geq 0$, $||\sigma||^2 = h^{ab}h^{cd}\sigma_{ac}\sigma_{bd} \geq 0$ are transverse norms,
signs indicating that Θ satisfies DEC: gravitational radiation carries positive energy.
The components may be interpreted as gravitational energy densities,
by geodesic deviation of test particles [Szekeres 1965]:
Θ_{++}: ingoing transverse mode, $\Theta_{++} \to$ Bondi energy density at past null infinity,
Θ_{-+}: ingoing longitudinal mode, $r^2\Theta_{-+} \to 0$ at null infinity,
Θ_{+-}: outgoing longitudinal mode, $r^2\Theta_{+-} \to 0$ at null infinity,
Θ_{--}: outgoing transverse mode, $\Theta_{--} \to$ Bondi energy density at future null infinity.
The $\Theta_{\pm\pm}$ components also recover expressions for energy density of gravitational radiation,
with $\Theta_{\pm\mp}$ vanishing, in the high-frequency linearized approximation [Isaacson 1968],
in cylindrical symmetry [SAH 2000] and in a quasi-spherical approximation [SAH 2001].

5. Conservation of angular momentum [SAH 2006, cf. Ashtekar & Krishnan 2002]



The standard definition of angular momentum [Komar 1959]

for an axial Killing vector ψ and at spatial infinity is

$$J[\psi] = -\frac{1}{16\pi}\oint_S *\epsilon_{\alpha\beta}\nabla^\alpha\psi^\beta, \quad \text{where } \epsilon_{AB} \text{ is the binormal.}$$

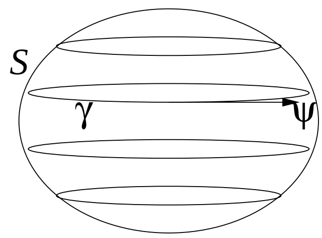
For a general transverse vector ψ, $\perp\psi = \psi$, the Komar integral can be rewritten as

$$J[\psi] = \frac{1}{8\pi}\oint_S *\psi^a\omega_a$$

where the twist $\omega_a = \frac{1}{2}e^f h_{a\beta}[l_-, l_+]^\beta$ is a transverse ($\perp\omega = \omega$) 1-form, measuring the non-integrability of the normal space.

For the weak-field metric in spherical polar coordinates $(t, r, \vartheta, \varphi)$ [MTW], $J[\partial_\varphi]$ recovers the standard definition of angular momentum, with the precessional angular velocity $(\vec{\Omega}\cdot\hat{r})\hat{r} - \frac{1}{3}\vec{\Omega}$ of a gyroscope in the unit direction $\hat{r}$, due to the Lense-Thirring effect, directly related to the twist by $\omega \sim \vec{\Omega}\times\hat{r}$.
Thus the twist indeed encodes the twisting around of space-time due to a rotating mass.

The twist is an invariant of a non-null foliated hypersurface H, so the twist expression for $J[\psi]$ is also an invariant of H.
The twist coincides with the 1-form used for dynamical horizons [Ashtekar & Krishnan 2002] to define angular momentum, but not that used for isolated horizons [Ashtekar et al. 2000], which is $\omega + \frac{1}{2}Df$, where D is the covariant derivative of h.
They will give compatible $J[\psi]$, by the Gauss divergence theorem, if the axial vector ψ has vanishing transverse divergence, $D_a\psi^a \cong 0$ [Ashtekar & Krishnan 2003].
If there exist angular coordinates (ϑ, φ) on S, completing coordinates (x, ϑ, φ) on H, such that $\psi = \partial_\varphi$, then recalling that $\xi = \partial_x$ and that coordinate vectors commute, $L_\xi\psi \cong 0$, previously proposed as a natural way to propagate ψ along ξ [Gourgoulhon 2005].
The commutator identity $L_\xi(D_a\psi^a) - D_a(L_\xi\psi)^a = \psi^a D_a\theta_\xi$ then yields $\psi^a D_a\theta_\xi \cong 0$.
This is automatic if $D\theta_\xi \cong 0$, as in spherical symmetry or along a null trapping horizon.
However, generically one expects $D\theta_\xi \not\cong 0$ almost everywhere.



The hairy ball theorem states that a continuous vector field,

e.g. $D^a\theta_\xi$, must vanish somewhere on a sphere;

however, the simplest generic situation is that the curves γ

of constant θ_ξ form a smooth foliation of circles with two poles.

Assuming so, ψ must be tangent to γ.

Then one can find a unique ψ, up to sign, in terms of the unit tangent vector $\hat{\psi}$

and arc length ds along γ: $\psi \cong \hat{\psi} \oint_\gamma ds/2\pi$, so that φ is identified at 0 and 2π.

Then the angular momentum becomes unique up to sign, $J[\psi] = J$.

The sign is naturally fixed by $J \geq 0$ and continuity of ψ, reflecting a choice of orientation.

For an axisymmetric space-time with axial Killing vector ψ, $D_a\psi^a = 0$,

while if ξ respects the symmetry, $0 = L_\psi \xi = -L_\xi \psi$,

so the above construction, if unique as assumed, yields the correct ψ.

For a Kerr space-time in Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$,

with surfaces given by constant (t, r) and $\xi = \partial_r$, the construction yields $\psi = \partial_\varphi$.

Conservation of angular momentum $L_\xi J \cong -\oint_S *(T_{aB} + \Theta_{aB})\psi^a \tau^B$

holds along a trapping horizon under the above conditions $L_\xi \psi \cong 0$, $D_a\psi^a \cong 0$, where

$$\Theta_{a\pm} = -\frac{1}{16\pi} h^{cd} D_d \sigma_{\pm ac}$$

is the transverse-normal block of the effective energy tensor for gravitational radiation.

Recalling the energy densities $\Theta_{\pm\pm} = ||\sigma_\pm||^2/32\pi$,

indicating that transverse gravitational radiation is encoded in null shear $\sigma_{\pm ab}$,

it seems that differential gravitational radiation has angular momentum density.

So this describes how a black hole spins up or down,

due to infall of co-rotating or counter-rotating matter or gravitational radiation.

Thus conservation of energy and angular momentum take a similar form:

$L_\xi M \cong \oint_S *(T_{\alpha\beta} + \Theta_{\alpha\beta}) k^\alpha \tau^\beta$

$L_\xi J \cong -\oint_S *(T_{\alpha\beta} + \Theta_{\alpha\beta}) \psi^\alpha \tau^\beta$.

Both take the same form as standard expressions in flat space-time

for a stationary Killing vector k and an axial Killing vector ψ,

except for the inclusion of gravitational radiation in Θ.

They are the independent conservation laws expected for an astrophysical black hole,

which defines its own centre-of-mass frame, in which its momentum vanishes.

6. Quasi-local conservation laws [SAH 2006]

Charge Q is defined in terms of charge-current density j as

$$[Q] = -\int_H *g(j,\tau) \wedge dx = -\int_H \hat{*}g(j,\hat{\tau})$$

where the second (more usual) expression holds only for a spatial hypersurface H.

The surface-integral form is

$$L_\xi Q = -\oint_S *g(j,\tau).$$

The above conservation laws can be written in the same form

$$L_\xi M \cong -\oint_S *g(\bar{\jmath},\tau), \quad L_\xi J \cong -\oint_S *g(\tilde{\jmath},\tau)$$

by identifying current vectors

$$\bar{\jmath}^B = -k_A(T^{AB} + \Theta^{AB}), \quad \tilde{\jmath}^B = \psi_a(T^{aB} + \Theta^{aB}).$$

The standard physical interpretation of the conserved vectors is

$\bar{\jmath} = (\text{energy density}, \text{energy flux})$,

$\tilde{\jmath} = (\text{angular momentum density}, \text{angular stress})$,

$j = (\text{charge density}, \text{current density})$.

For spatial ξ,

$\oint_S *g(\bar{\jmath},\xi) = \text{power}, \ -\oint_S *g(\bar{\jmath},\tau) = \text{gradient of energy}$,

$\oint_S *g(\tilde{\jmath},\xi) = \text{torque}, \ -\oint_S *g(\tilde{\jmath},\tau) = \text{gradient of angular momentum}$,

$\oint_S *g(j,\xi) = \text{current}, \ -\oint_S *g(j,\tau) = \text{gradient of charge}$.

Charge conservation in local differential form is $\nabla_\alpha j^\alpha = 0$.

For energy and angular momentum, one has only quasi-local conservation laws:

$$\oint_S *\nabla_\alpha \bar{\jmath}^\alpha \cong \oint_S *\nabla_\alpha \tilde{\jmath}^\alpha \cong 0.$$

It seems that energy and angular momentum in General Relativity can be quasi-localized: defined as surface integrals [Penrose 1982].

7. State space [SAH 2006]

There are now three conserved quantities (M, J, Q), as for a Kerr-Newman black hole. One can use the Kerr-Newman formula for the ADM energy to define a mass-energy

$$E \cong \frac{\sqrt{((2M)^2 + Q^2)^2 + (2J)^2}}{4M}$$

for each marginal surface in a trapping horizon, where $R \cong 2M$, and other Kerr-Newman formulas to define surface gravity

$$\kappa \cong \frac{(2M)^4 - (2J)^2 - Q^4}{2(2M)^3\sqrt{((2M)^2 + Q^2)^2 + (2J)^2}},$$

angular speed

$$\Omega \cong \frac{J}{M\sqrt{((2M)^2 + Q^2)^2 + (2J)^2}}$$

and electric potential

$$\Phi \cong \frac{((2M)^2 + Q^2)Q}{2M\sqrt{((2M)^2 + Q^2)^2 + (2J)^2}}.$$

Then $E \geq M$ can be interpreted as the effective mass of the black hole, including irreducible mass M, rotational kinetic energy $\approx \frac{1}{2}I\Omega^2$ and electrostatic energy $\approx \frac{1}{2}Q^2/R$, by expanding $E \approx M + \frac{1}{2}I\Omega^2 + \frac{1}{2}Q^2/R$ for $J \ll M^2$ and $Q \ll M$, where $J = I\Omega$ defines the moment of inertia $I \cong M\sqrt{((2M)^2 + Q^2)^2 + (2J)^2} \cong ER^2$.

State-space formulas

$$\kappa \cong 8\pi\frac{\partial E}{\partial A} \cong \frac{1}{4M}\frac{\partial E}{\partial M}, \quad \Omega \cong \frac{\partial E}{\partial J}, \quad \Phi \cong \frac{\partial E}{\partial Q}$$

then yield a dynamic version

$$L_\xi E \cong \frac{\kappa}{8\pi}L_\xi A + \Omega L_\xi J + \Phi L_\xi Q$$

of the "first law of black-hole mechanics".

8. Equilibrium: null trapping horizons [SAH 2006]

A trapping horizon is in local equilibrium when it becomes null,

e.g. when a growing black hole ceases to grow.

Zeroth law: DEC $\Rightarrow g(\bar{\jmath},\tau)\cong g(\tilde{\jmath},\tau)\cong g(\jmath,\tau)\cong 0$ along a null trapping horizon

$\Rightarrow (M,J,Q)$ constant $\Rightarrow (E,\kappa,\Omega,\Phi)$ constant.

This is stronger than the zeroth law of thermodynamics,

since it expresses complete (not just thermal) equilibrium.

Thus, by the area law (which includes $L_\xi A\cong 0\Rightarrow H$ null),

a black hole cannot change its angular momentum or charge without increasing its area.

On a null trapping horizon, one may take $\xi\cong\tau\cong l_+$,

but the other null vector l_- is non-unique, leading to non-uniqueness in ω,

used to define angular momentum here and for dynamical horizons [Ashtekar & Krishnan 2002].

However, $\omega+\frac{1}{2}Df$ is unique, an intrinsic normal fundamental form of a null hypersurface,

therefore used to define angular momentum for isolated horizons [Ashtekar et al. 2000].

On the other hand, the extrinsic normal fundamental form $\omega-\frac{1}{2}Df$ is preserved:

DEC $\Rightarrow L_+(\omega-\frac{1}{2}Df)\cong 0$.

The non-uniqueness can be fixed by $Df\cong 0$, or $Dg(l_+,l_-)\cong 0$.

Then all three normal fundamental forms coincide.

Also $L_\xi\omega\cong 0$, so that $L_\xi J\cong 0$ assuming only that ψ is a coordinate vector.

Comparing with energy, NEC $\Rightarrow L_\xi M\cong 0$ automatically on a null trapping horizon.

Thus consideration of angular momentum resolves an ambiguity in taking the null limit.

This has largely recovered the notion of weakly isolated horizon [Ashtekar et al. 2000]

except that the (allowable) scaling freedom in ξ has not been fixed.

9. Summary

Laws of black-hole dynamics

Conservation of energy: $L_\xi M \cong \oint_S *(T_{AB} + \Theta_{AB})k^A\tau^B$

Conservation of angular momentum: $L_\xi J \cong -\oint_S *(T_{aB} + \Theta_{aB})\psi^a\tau^B$

"Zeroth": $(M, J, Q, E, \kappa, \Omega, \Phi)$ constant on null trapping horizons (DEC)

"First": $L_\xi E \cong \kappa L_\xi A/8\pi + \Omega L_\xi J + \Phi L_\xi Q$

Area ("second"): $L_\xi A \geq 0$, for future outer trapping horizons (NEC)

Signature: achronal, for outer horizons (NEC)

Topology: spherical marginal surfaces, for future outer trapping horizons (DEC)

+ versions for other horizon types and energy conditions:
evaporating black holes, white holes, traversable wormholes, cosmological horizons...

Themes

Local or quasi-local analysis, cf. asymptotics

Dynamics vs. statics (general vs. stationary space-times)

Geometry: covariant quantities $(*1, A, H, \xi, \tau, k, \psi)$ and equations

Physics: physically meaningful quantities $(M, J, Q, \Theta, E, \kappa, \Omega, \Phi)$ and equations

Black holes, gravitational radiation, energy and angular momentum in General Relativity, three subjects which are often regarded as difficult or impossible to understand in general, do make sense quite generally and are fundamentally interrelated.