Motion of Isolated bodies

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Quite general initial data for massive isolated bodies in General Relativity can be constructed by the conformal method. The problem of their evolution is more difficult to handle. The Euler equations for fluid bodies become singular in regions where the density vanishes, except for some very special equations of state discovered by Rendall, but, as he points out himself, his solution is not unique.

We show that when the pressure can be neglected the Einstein - matter system with initial data massive isolated bodies form a well posed system, whose solution is physically unique.

Einstein - dust equations

$$S_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T^{\lambda}_{\lambda},$$

∇ covariant differential in the spacetime metric g.
Conservation law

$$\nabla_{\alpha} T^{\alpha\beta} = 0.$$

Pressure free matter (dust)

$$T_{\alpha\beta} = r u_{\alpha} u_{\beta},$$

r scalar matter density, u future directed flow vector field,

$$u^{\alpha} u_{\alpha} = -1, \text{ hence } \quad u^{\alpha} \nabla_{\beta} u_{\alpha} = 0,$$

Projection of conservation law orthogonal to u,

$$r\, u^\alpha \nabla_\alpha u_\beta = 0 \;\; \rightarrow \quad u^\alpha \nabla_\alpha u_\beta = 0 \;\; \text{if} \;\; r \neq 0,$$

geodesic equation.

Projection into the direction of u gives continuity equation

$$\nabla_\alpha (r u^\alpha) = u^\alpha \partial_\alpha r + r \nabla_\alpha u^\alpha = 0.$$

In $n+1$ dimensional spacetime Einstein equations with dust source read

$$R_{\alpha\beta} = r(u_\alpha u_\beta + \frac{1}{n-1} g_{\alpha\beta}).$$

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Dynamics.

Unknowns: g, r, u.

Initial data on a smooth n dimensional manifold M.

Geometric data: Riemannian metric $\bar{g}$, symmetric 2-tensor $\bar{k}$.

Matter data: scalar function $\bar{r}$, tangent vector field $\bar{v}$ to M, $\bar{u}$ is the spacetime unit vector whose orthogonal projection on M is $\bar{v}$, i.e.

$$\bar{N}^{-2}(\bar{u}_0)^2 - \bar{g}^{ij}\bar{u}_i\bar{u}_j = 1, \ \bar{u}_i = \bar{v}_i,$$

$\bar{u}_0$ depends on the choice $\bar{N}$ of the lapse on M.

Einstein - dust development (V, g, r, u) of $(M, \bar{g}, \bar{k}, \bar{r}, \bar{v})$: solution of the Einstein - dust equations such that M identified with an embedded submanifold of (V, g), $\bar{g}$ and $\bar{k}$ are the induced metric and extrinsic curvature on M, $\bar{r}$ is the function induced by r on M and $\bar{v}$ is the projection on M of $\bar{u}$, value of u on M.

Constraints.

Momentum constraint

$$\bar{\nabla}.\bar{k} - \bar{\nabla} tr \bar{k} - \bar{N}^{-1}\bar{r}\bar{u}_0\bar{v} = 0,$$

Hamiltonian constraint

$$R(\bar{g}) - |\bar{k}|^2_{\bar{g}} + (tr_{\bar{g}}\bar{k})^2 - 2\bar{N}^{-2}\bar{r}(\bar{u}_0)^2 = 0,$$

Depend only on initial data

Conformal method, $\tau = Trace\bar{k}$, $\tilde{g}$, $\tilde{k}$ given.

$$\bar{g}_{ij} = \varphi^{\frac{4}{n-2}}\tilde{g}_{ij}, \quad \bar{k}_{ij} = \varphi^{-2}\tilde{k}_{ij} + \frac{1}{n}\bar{g}_{ij}\tau.$$

$\tilde{u}, \tilde{r}$? $\tilde{N}$?

$$\tilde{N}^{-2}(\tilde{u}_0)^2 - \tilde{g}^{ij}\tilde{u}_i\tilde{u}_j = 1.$$

satisfied, for any choice of $\tilde{N}$, by

$$\tilde{u}_i = \varphi^{-\frac{2}{n-2}}\bar{u}_i, \quad \bar{N}^{-1}\bar{u}_0 = \tilde{N}^{-1}\tilde{u}_0.$$

The momentum $\bar{J}$ reads then

$$\bar{J}_i := \bar{N}^{-1}\bar{r}\bar{u}_0\bar{u}_i = \varphi^{\frac{2}{n-2}}\tilde{N}^{-1}\tilde{u}_0\tilde{u}_i\bar{r}.$$

This momentum is York scaled if $\tilde{r}$ is defined by

$$\bar{r} = \varphi^{\frac{-2(n+1)}{n-2}}\tilde{r}.$$

Physical interpretation: equality on M of spacetime densities

$$\bar{r}\bar{N}det\bar{g}^{\frac{1}{2}} = \tilde{r}\tilde{N}(det\gamma)^{\frac{1}{2}}$$

if $\bar{N}^2 = \varphi^{\frac{4}{n-2}}\tilde{N}^2$, same factor as $\bar{g}$.
Source of Hamiltonian constraint then York scaled:

$$\bar{N}^2\bar{T}^{00} \equiv \bar{r}(\bar{N}\bar{u}^0)^2 = \varphi^{\frac{-2(n+1)}{n-2}}\tilde{r}(\tilde{N}\tilde{u}^0)^2.$$

The constraints decouple and known theorems apply.

Evolution.

Theorem. The Einstein - dust equations in wave gauge form a hyperbolic Leray system for g, u and r, which is causal as long as g is Lorentzian and u is timelike.

Proof. The equations are: $R^{(h)}_{\alpha\beta} \equiv$

$$-\frac{1}{2}g^{\lambda\mu}\nabla_\rho\partial^2_{\lambda\mu}g_{\alpha\beta} + H_{\alpha\beta}(g, \partial g) = r(u_\alpha u_\beta + \frac{1}{n-1}g_{\alpha\beta}) \qquad ((1))$$

$$u^\alpha\nabla_\alpha u_\beta = 0, \qquad ((2))$$

$$u^\alpha\partial_\alpha r + r\nabla_\alpha u^\alpha = 0. \qquad ((3))$$

Replace (1) using (2) by

$$u^\rho\nabla_\rho R^{(h)}_{\alpha\beta} = (u^\rho\nabla_\rho r)(u_\alpha u_\beta + \frac{1}{n-1}g_{\alpha\beta}) \qquad ((4))$$

Give to equations and unknowns the Leray - Volevic indices

$$\ell(g) = 3, \quad \ell(u) = 2, \quad \ell(r) = 1.$$

$$m(4) = 0, \quad m(2) = 1, \quad m(3) = 0,$$

The matrix of principal parts of the various orders $\ell - m$ is then diagonal and given by

$$\begin{pmatrix} -\frac{1}{2} g^{\lambda\mu} u^{\gamma} \partial^3_{\gamma\lambda\mu} g_{\alpha\beta} & 0 & 0 \\ 0 & u^{\alpha} \partial_{\alpha} u^{\beta} & 0 \\ 0 & 0 & u^{\alpha} \partial_{\alpha} r \end{pmatrix}.$$

Hyperbolic system if g is Lorentzian and u timelike.

Corollary. One and only one solution, g in wave gauge, globally hyperbolic, development of the initial data set $(M,\ \bar{g} \in M_s,\ \bar{k} \in H^{loc}_{s-1},\ \bar{v} \in H^{loc}_{s-1}$, $|\bar{v}|_{\bar{g}} < 1,\ \ \bar{r} \in H^{loc}_{s-2})$, $s > \frac{n}{2} + 2$, lapse, shift and first derivatives given on M such that the wave gauge conditions are satisfied initially.

Bianchi identities imply that a solution in wave gauge, satisfy the original Einstein - dust system if the initial data satisfy the Einstein constraints.

Unicity, up to diffeomorphism, in the class of maximal globally hyperbolic spacetimes if $s > \frac{n}{2} + 3$. Remark that the vectors $\bar{u}$ and $\bar{u}_*$ corresponding to the same $\bar{v}$ and two spacetime metrics isometric under a diffeomorphism f are mapped onto each other by f..

Case of isolated bodies.

Suppose that $\bar{r}$ vanishes on $M\backslash\omega$ where ω is the union of disjoint relatively compact subsets of M which represent the space occupied by material bodies. While $\bar{v}$ represents the physically well defined 'matter flow velocity' at points of ω, it has no physical meaning at points of $M\backslash\omega$. The following theorem says that neither g, nor r depend on the extension of $\bar{v}$ to M, the same is true of u in the geodesic tube, containing the support of r, based on ω.

Theorem. The spacetime metric g in wave gauge, the matter density r and its support Ω, obtained in the existence theorem are uniquely determined in a neighbourhood of M by

data on M of $\bar{g}$, $\bar{k}$, lapse and shift and their first derivatives, and of $\bar{r}$

datum $\bar{v}$ on ω with $\bar{\omega} = supp(\bar{r})$.

The same is true for the vector field u but only in Ω.

Proof. Suppose, $\bar{u}_*$ is another datum for u on M, with $\bar{u}_* = \bar{u}$ on ω. The datum $\bar{g}, \bar{k}, \bar{v_*}, \bar{r}$ determine a solution g_*, u_*, r_* in a neighbourhood V_* of M. Consider the geodesics in the original metric g starting from M with tangent vector $\bar{u}_*$. Since $\bar{u}_* = \bar{u}$ on ω, and the geodesic equation depends only on g, $u_* = u$ in the geodesic tube Ω. The continuity equation, with u replaced by u_* and $\bar{r} = \bar{r}_*$ on M, determines r_*; the functions r and r_* have their support in Ω, and coincide there, hence $r_* = r$ on V.

The right hand side of Einstein equations unchanged if we replace u by u_*, since $r = 0$ where $u \neq u_*$. The fields g, u_*, r thus define on $V_* := V \cap U_*$, with U_* the domain of existence of u_*, a solution with the star initial data, by the uniqueness theorem this solution coincides with the solution g_*, u_*, r_* . Therefore $g = g_*$ and $r_* = r$ on V_*. We have already seen that $u = u_*$ on Ω.

Remark 1. The Einstein - dust constraints on M do not depend on the extension of $\bar{v}$ at points where $\bar{r}$ vanishes.

Remark 2 It is not expected that the domain of existence of the vector field u_* outside of Ω is the same for all possible extensions of $v|_\omega$ to M. This gauge choice may perhaps be used to optimize the domain of existence of the solution.